High frequency radiation from dynamic earthquake fault models

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ABSTRACT. We study the radiation of high frequency waves from a simple antiplane model of an earthquake source. In this model only antiplane waves are generated so that the mathematics is relatively simple, but the physics is the same as in the more complex plane or three dimensional models where P and S waves are radiated. An exact solution is found for the problem of an arbitrarily moving semi-infinite crack in the presence of a general dynamic stress drop.

In the case when friction is independent of time, an algebraic expression is obtained for particle velocity. This result is exploited to understand the origin of high frequency waves, and the role of rupture velocity and stress intensity on the radiation. We show that barriers and asperities dominate the radiation, but that they are indistinguishable from a high frequency point of view.

Key words : seismology, seismic sources, elastic wave propagation, brittle fracture.

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INTRODUCTION

Understanding the generation of high frequency waves during earthquake faulting is essential for the prediction of strong motion. A number of recent observations have demonstrated that high frequency acceleration may be modelled as a finite-duration, band limited, white Gaussian noise (Hanks and Mc Guire, 1981). The acceleration spectra are flat at high frequency limited at the lower end by the corner frequency and at the high end by either attenuation, instrument response or, perhaps some geometrical properties of the source. A physical interpretation of this result requires an understanding of the process of generation of high frequencies during earthquake faulting.

The usual approach to model high frequency waves in the near field is to represent the source as a superposition of point sources. The problem reduces then to specifying the source time function (slip velocity) at every point. This approach allows for the kinematic description of rupture velocity, barriers, and asperities, but it is not clearly related to the dynamics of faulting; and it becomes very expensive at high frequency since a large number of points on the fault becomes necessary to maintain accuracy. Furthermore it is possible to show for simple dislocation models that the radiation from the different point sources interferes destructively except near the borders of the dislocation (Madariaga, 1978). Thus, a detailed study of high frequency generation may lead to simpler and cheaper methods of modelling and interpreting near field strong motion. The usual way to model high frequency radiation has been to build random models of the slip velocity on the fault and then to calculate the far field radiation (Haskell, 1964; Aki, 1967; Andrews, 1981). In this approach the underlying stress release mechanism is not taken into account, nor is the causal spreading of the rupture front during faulting.

A different approach, based on dynamical fracture mechanics, was adopted by Madariaga (1977) and Achenbach and Harris (1978). Noting that the radiation from a fault is entirely controlled by the slip velocity field in the ruptured portion of the fault, they proposed that slip velocity has a number of universal topological features that should be incorporated into any model of high frequency generation. The most important property is that slip velocity is strongly concentrated behind the rupture front. Even in the presence of barriers, asperities, multiple sources or other complexities on the fault these strong slip velocity concentrations are always there. It is then proposed that the radiation of high frequency waves is controlled by the motion of the slip velocity concentrations.

Barriers and asperities produce large variations of the intensity of these concentrations and are the source of the high frequency waves. In this paper we will present a complete solution for the radiation of high frequency waves from antiplane two dimensional cracks in the presence of asperities and barriers. The extension of these results to three dimensions is possible, provided that a few canonical problems can be solved. We propose that in three dimensions, high frequency waves are generated by the motion of the rupture front, its stopping, acceleration and eventual disappearance at the free surface. This leads to a model where the source of high frequencies at any instant of time is a curved line coinciding with the rupture front. The shape of this line is very general, it may be open or closed. For this reason and to stress its main topological feature, we call this model the string model of high frequency radiation.

THE ANTIPLANE CRACK MODEL

In order to establish the basic physics of the radiation of high frequencies we choose the simplest possible fault configuration : a two dimensional antiplane crack (fig. 1). In this model slip occurs only in the y direction

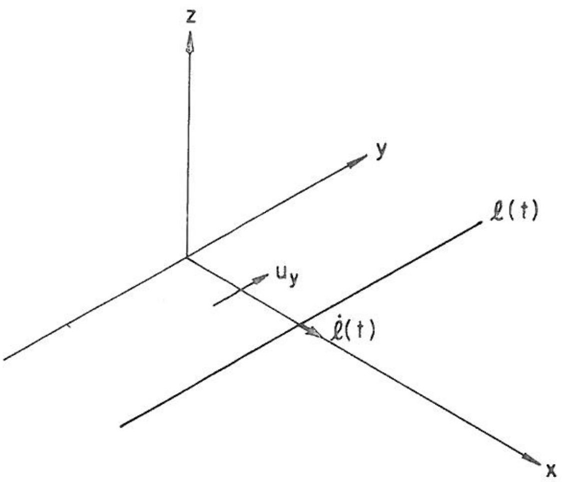

Figure 1
Geometry of the antiplane crack problem. The rupture front moves along the x direction, its position as a function of time is given by $\ell(t)$ and its instantaneous rupture velocity by $\dot{\ell}(t)$. Slip on the ruptured part of the fault plane is in the y direction (antiplane or SH).

and the rupture fronts are infinite straight lines parallel to the y-axis. Only SH waves are generated which simplifies enormously the analytical work. Solutions for the plane problem also exist but the basic physics is entirely contained in the simpler antiplane problem. In this two dimensional model the high frequencies originate from the rupture front which is a straight line, i.e. we have a straight string source.

Consider the geometry shown in figure 1 : an antiplane rupture moves with an arbitrary rupture velocity $\dot{\ell}$ along the x axis. The initial state of stress is a pure shear stress $\sigma^0_{yz}(x, z)$. Inside the crack, after the passage of the rupture the stress drops to the dynamic friction $\sigma^f_{yz}(x)$. The difference :

$$\sigma_e(x) = \sigma^0_{yz}(x, 0) - \sigma^f_{yz}(x) \qquad (1)$$

is the dynamic stress drop, that is the stress that is available to drive the slip on the crack and generate seismic waves. The solution to the general problem of determining the slip velocity for arbitrary motion of the crack tip and heterogeneous stress drop was obtained in the classical work by Kostrov (1966) and discussed in detail by Aki and Richards (1980, p. 884). For a finite crack the solution leads to a multiple diffraction problem by the tips of the crack. For simplicity we shall consider here a semi infinite crack extending along the axis $x_1 < \ell(t_1)$, where $\ell(t_1)$ is the current position of the crack tip. This is sufficient to model the dominating high frequency waves radiated by cracks since multiple diffracted waves are significantly weaker than direct phases.

The problem is to find the velocity and stress field on the half space $z > 0$ for the boundary conditions on the plane $z = 0$:

$$\begin{aligned} \sigma_{yz}(x, 0, t) &= \sigma_e(x, t) \qquad & -\infty < x < \ell(t) \\ u_y(x, 0, t) &= 0 & x > \ell(t) \end{aligned} \qquad (2)$$

where σ_e is the stress drop assumed to be a function of time and position on the crack. u_y is the displacement on the y-direction, all other components of displacement are identically zero (SH problem).

The solution for the stress σ_{yz} outside the crack was found by Kostrov (1966) to be :

$$\sigma_{yz}(x, 0, t) = -\frac{1}{\sqrt{x - \ell(\tau)}} \int_{x-\beta t}^{\ell(\tau)} \times \times \sigma_e\left(x_1, t - \frac{x - x_1}{\beta}\right) \frac{\sqrt{\ell(\tau) - x_1}}{x - x_1} dx_1 \qquad (3)$$

for $x > \ell$ where $\ell(\tau)$ is the solution of :

$$x - \ell(\tau) = \beta(t - \tau)$$

and β is the shear velocity. $\ell(\tau)$ is the position of the crack tip when the wave reaching x at time t was emitted. $\ell(t_1)$ is the position of the crack tip as a function of time. We assume that $\ell(t_1)$ is a given monotonically increasing function of time. It may be calculated by a rupture criterion as explained by Kostrov (1966). We assume here that the rupture velocity $\dot{\ell}(t_1)$ is always subsonic. The integration in equation (3) is illustrated in figure 2.

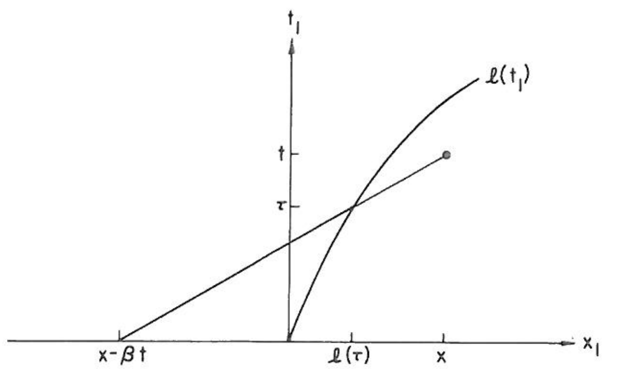

Figure 2
Line of integration in the (x_1, t_1) plane for the stress (eq. (3)) on the fault plane outside the crack. The rupture front position as a function of time is given by the curve $\ell(t_1)$. The intersection of the backward characteristic through the point of calculation (x, t) and the rupture front position is the retarded position of the rupture front $[\ell(\tau), \tau]$.

We have now the complete solution for the stress on the line $z = 0$; $\sigma_e(x_1, t_1)$ for $x_1 < \ell(t_1)$ and $\sigma_{yz}(x_1, 0, t_1)$ for $x_1 > \ell(t_1)$. The displacement field $u_y(x, z, t)$ inside the half plane $z > 0$ may be calculated by the representation theorem :

$$u_y(x, z, t) = \frac{\beta}{\pi\mu} \iint_S \frac{\sigma_{yz}(x_1, 0, t_1)}{R} dx_1\, dt_1 \qquad (4)$$

where :

$$R = (\beta^2(t - t_1)^2 - (x - x_1)^2 - z^2)^{1/2} \qquad (5)$$

and μ and β are the rigidity and shear velocity, respectively. The domain of integration in the (x_1, t_1) plane is indicated in figure 3 by the area $S_1 + S_2 + S_3$. To the left of the line $\ell(t_1)$ it integrates over the known stress drop, while to the right we have to use the stress calculated in (3). Following some results by Slepjan (1980) we found that the integral in (4) may be simplified once we replace the value of σ_{yz} outside the crack, $x_1 > \ell(t_1)$, by the expression (3). After some algebra we find that the integral $S_2 + S_3$ is exactly zero, so that

$$u_y(x, z, t) = \frac{\beta}{\pi\mu} \iint_{S_1} \frac{\sigma_e(x_1, t_1)}{R} \, dx_1 \, dt_1 \qquad (6)$$

where the area of integration S_1 is shown in figure 3. The expression (6) is an integral over the stress drop σ_e *inside* the crack only, this is an extraordinarily simple result. The proof of (6) is given in Appendix I. The equivalent of (6) for the slip on the crack ($z = 0$) was obtained by Ida (1973).

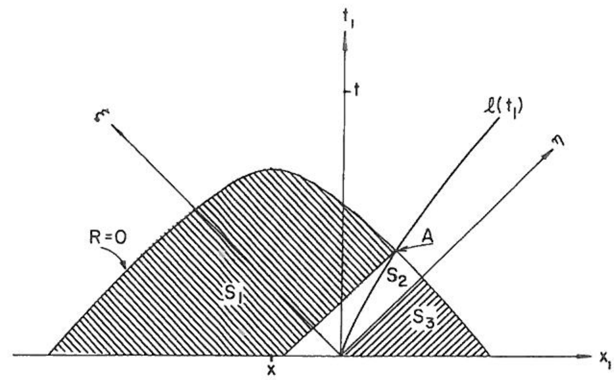

Figure 3

The surface of integration on the (x_1, t_1) plane for the evaluation of equations (4) and (6) for the displacement at a given point and time (x, z, t). The surface is bounded above by the hyperbola given by equation (5) with $R = 0$. A is the intersection of this hyperbola with the rupture front position; its coordinates are given by the solution $[\ell(\tau), \tau]$ of equation (7). On Appendix I it is shown that the integrals over the surfaces S_2 and S_3 vanish identically.

The integral in (6) may now be interpreted. As seen from figure 3 the displacement u_y at time t contains information about the crack tip only from point A. The previous positions of the crack tip do not affect at all the result. This property was first pointed out by Eshelby (1969) for a particular case and is extended here to arbitrary loading of the crack. The point A is defined as the retarded position of the crack, that is, it is the position of the rupture front when the waves reaching the point (x, z) at time t were emitted.

The retarded time τ and position $\ell(\tau)$ are given by :

$$\beta(t - \tau) = [(x - \ell(\tau))^2 - z^2]^{1/2} = R^1(\tau). \qquad (7)$$

Given the rupture front position $\ell(t)$ as a function of time, τ and $\ell(\tau)$ are solved from (7); see also figure 4. A general discussion about the solution of this equation for subsonic and supersonic moving sources was given by Freund (1972). The distance to the rupture front R^1 at the time of emission of the wave will play a fundamental role in the results we are going to present in the following.

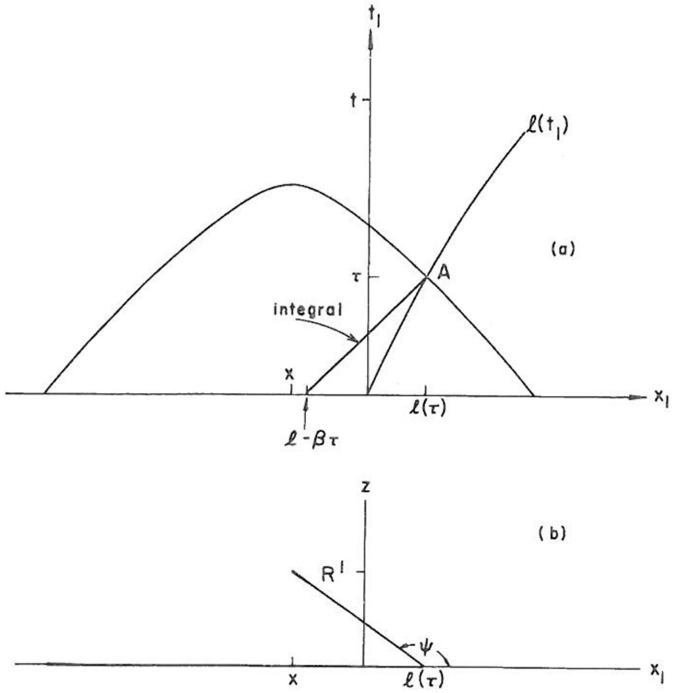

Figure 4

Geometry of the (x_1, z, t_1) plane for the evaluation of particle velocity at the given point (x, z, t). The point A given by $[\ell(\tau), 0, \tau]$ is the retarded position of the rupture front when the radiation reaching the observation point was emitted. The integral in (9) is taken along the backward characteristic through A. At the bottom (b) we picture the geometrical definition of R^1, the distance from the observation point to the retarded position of the crack tip. ψ is the angle of radiation measured from the positive x-axis.

The properties of the radiation are even clearer when we consider the particle velocity $v_y = \partial u_y/\partial t$. This may be obtained by differentiation of (6) with respect to time. Two terms appear : one is the same integral as (6) but with $\sigma(x, t)$ replaced by $\partial\sigma(x, t)/\partial t$, the rate of stress change. During an earthquake the dynamic stress drop does not change significantly with time at a given position on the fault, i.e. we assume $\sigma_e(x, t) = \sigma_e(x)$. In this case the integral over time in (6) may be evaluated exactly, and taking the time derivative we find (see Appendix II) :

$$v_y(x, z, t) = \frac{1}{\pi\mu} \int \frac{\sigma_e(x_1)}{\sqrt{q^2 - (x - x_1)^2 - z^2}} \frac{dq}{dt} dx_1 \qquad (8)$$

where :

$$q = (\ell(\tau) - x_1) + R^1$$

and $\ell(\tau)$ is the retarded position. Calculating the derivative $\dot{q}$ is straightforward :

$$\dot{q} = dq/dt = \frac{R^1 - (x - \ell(\tau))}{[R^1 - \dot{\ell}/\beta(x - \ell(\tau))]} \cdot \dot{\ell}$$

Let us note that $\dot{q}$ does not depend on x_1 ! Reducing the square root in (8) a little further we finally find that :

$$v_y(x, z, t) = \frac{\dot{\ell}}{\mu} \frac{\sqrt{[R^1 - (x - \ell(\tau)]/2}}{R^1 - \dot{\ell}/\beta(x - \ell(\tau))} \int^{\ell(\tau)} \frac{\sigma_e(x_1)\,dx_1}{\sqrt{\ell(\tau) - x_1}} \tag{9}$$

where $\dot{\ell}(\tau)$ is the rupture velocity at the retarded time τ which is calculated, together with $\ell(\tau)$ from (7). The significance of the result (9) is clear : we may calculate the velocity anywhere in the medium by a single integral of the stress drop on the crack. The integration on the (x, t) plane is shown on the figure 4*a*. The connexion to the integral defining the stress outside the crack (3) is immediately obvious.

Let us note that (9) may be given a simpler form if one considers figure 4*b*. Here R^1 is the distance from (x, y) to the retarded position of the crack tip and ψ is the angle of radiation of the ray from the retarded position to the observer. Then,

$$v_y(x, z, t) = \frac{K_0(\ell)}{\mu} \dot{\ell} \frac{\sin \psi/2}{1 - \dot{\ell}/\beta \cos \psi} \frac{1}{\sqrt{R^1}} \tag{10}$$

where K_0 is the stress intensity of a static crack with its tip at $\ell(\tau)$:

$$K_0(\ell) = \frac{1}{\pi} \int_{\ell - \beta t}^{\ell} \frac{\sigma_e(x_1)\,dx_1}{\sqrt{\ell - x_1}}. \tag{11}$$

The result (10) has the same form as the first motion radiated by the sudden start or stop of an antiplane crack (Madariaga, 1977). But here it is much more general. Equation (10) is exact. It describes the entire velocity field for arbitrary motion of the crack tip. In particular, it contains the steady state solution for a semi infinite crack moving at constant velocity from infinity. Let us interprete it : at any given time the rupture front emits SH waves whose amplitude is proportional to the static stress intensity $K_0(\ell)$. These waves have a cylindrical decay $R^{-1/2}$, a directivity $(1 - \dot{\ell}/\beta \cos \psi)^{-1}$ and a radiation pattern $\sin \psi/2$. Equation (10) states that for static loading $\sigma(x, t) = \sigma(x)$ the *entire* field emanates from the crack tip, not just the high frequencies as assumed by Madariaga (1977). This is a surprisingly simple result when one considers the complexity of crack problems.

SEISMIC RADIATION AND STRESS INTENSITY

In order to clarify further the results (10), let us consider the properties of the stress and velocity fields in the vicinity of the crack tip, and their relationship with the radiated waves. The stress immediately outside the crack tip, $x \to \ell(t)$, may be calculated from eq. (3) :

$$\sigma_{yz}(x, 0, t) = \frac{1}{\sqrt{x - \ell(\tau)}} \frac{1}{\pi} \int_{x - \beta t}^{\ell(\tau)} \frac{\sigma_e(x_1)\,dx_1}{\sqrt{\ell(t) - x_1}} \tag{12}$$

where $\ell(\tau)$ is the retarded position of the crack tip. In order to introduce the current position of the rupture front $\ell(t)$ we use the following relationship, valid when $x \to \ell(t)$:

$$x - \ell(t) = (1 - \dot{\ell}/\beta)(x - \ell(\tau)).$$

Then :

$$\sigma_{yz}(x, 0, t) = K_d(x - \ell(\tau))^{-1/2} \qquad x > \ell(t) \tag{13}$$

with the dynamic stress intensity factor :

$$K_d = \sqrt{1 - \dot{\ell}/\beta}\, K_0. \tag{14}$$

Thus, K_0, defined in (11), is a factor of (14) that does not depend on the instantaneous value of the rupture velocity. K_0 depends only on the load and, for subsonic rupture, it has no information on the history of rupture. Thus if at time t, $\dot{\ell}(t)$ changes abruptly the dynamic stress intensity K_d does also change because of the factor $\sqrt{1 - \dot{\ell}/\beta}$. The separation of the rupture velocity dependent term from the load dependent term K_0 is valid even if the dynamic stress drop varies with time. In that case the definition of K_0 has to be modified slightly (Kostrov, 1966).

Let us examine now the velocity field in the vicinity of the crack tip. On the plane of the crack ($z = 0$), $\psi = \pi/2$ behind the rupture front, then :

$$v_y(x, 0, t) = \frac{K_0(\ell)}{\mu} \frac{\dot{\ell}}{1 + \dot{\ell}/\beta} (\ell(\tau) - x)^{-1/2}$$

and changing from the retarded position $\ell(\tau)$ to the current position of the crack tip, $\ell(t)$, we find asymptotically for $x \to \ell(t)$:

$$v_y(x, 0, t) = \frac{K_0(\ell)}{\mu} \frac{\dot{\ell}}{\sqrt{1 + \dot{\ell}/\beta}} (\ell(t) - x)^{-1/2} \tag{15}$$

and we may define the velocity intensity factor :

$$V_d(\ell) = \frac{K_0(\ell)}{\mu} \frac{\dot{\ell}}{\sqrt{1 + \dot{\ell}/\beta}} \tag{16}$$

which, just at K_d, separates into a load dependent factor and another one that depends only on the instantaneous velocity $\dot{\ell}$. Equation (15) shows that the velocity on the crack just behind the rupture front, has an inverse square root singularity of intensity V_d.

The amplitude of the SH waves defined by equation (10) is controlled by $K_0(\ell)$ the load dependent factor of the stress intensity and the instantaneous rupture velocity. We may say then that the elastic waves are generated by the motion of the stress intensity factor. When the crack tip stops moving, i.e. $\dot{\ell}(t) = 0$, the rupture front stops emitting seismic waves immediately, although a static stress intensity remains around the crack tip.

RADIATION OF HIGH FREQUENCY WAVES : BARRIERS AND ASPERITIES

We have obtained an expression (10) for the entire field radiated by the motion of the crack tip. How are high frequency waves generated ? If the crack tip moves smoothly with slowly varying rupture velocity and stress intensity K_0, the radiated waves will be also very smooth and long period. Strong high frequency radiation will be emitted only if either $\dot{\ell}$ or K_0 change rapidly. In most of the models studied in the literature the rupture front moved with constant velocity and was suddenly stopped. Most of the high frequency waves were then emitted during the sudden arrest of the crack; these are the so-called stopping phases. In complex models, like the barrier models of Das and Aki (1977), strong high frequency waves (acceleration pulses) were emitted every time the crack encountered a barrier. Another model of source complexity discussed in the literature is the asperity model, e.g. Rudnicki and Kanamori (1981). Depending on the strength of the asperity the strength of radiation will vary. Let us discuss both models in more detail.

The barrier model

A barrier was defined by Das and Aki (1977) as a region of increased rupture strength on the fault plane. Thus if a rupture moving along the fault encounters a barrier, it will either reduce rapidly its rupture velocity or, at a very strong asperity, stop completely. This will generate strong high frequency waves whose amplitude will be controlled by the jump in rupture velocity. K_0 does not change so that, if the position of the barrier on the fault plane is $\ell = \ell_0$, the radiation will be simply:

$$v_z(x, z, t) = \frac{K_0(\ell)}{\mu} \Delta\left[\frac{\dot{\ell}}{1 - \dot{\ell}/\beta \cos\psi}\right] \frac{\sin \psi/2}{\sqrt{R}} H(t - R/\beta) \tag{17}$$

the symbol Δ indicating a jump in the factor inside the brackets. This result was already found by Madariaga (1977) and in a slightly different form by Achenbach and Harris (1978). Therefore, a barrier produces a jump in rupture velocity which in turn produces a jump in the radiated field which is modulated by the directivity. The stopping phase is the limit case of the radiation by an unbreakable barrier, the rupture velocity drops to zero and the radiated wave will be proportional to $\dot{\ell}/(1 - \dot{\ell}/\beta \cos \psi)$ where $\dot{\ell}$ is the rupture velocity just before the crack encounters the barrier. This simple result may be verified in the seismograms calculated by Das and Aki (1977), although their radiation is not exactly like (17) because they took a slice of the two dimensional antiplane crack in order to simulate a finite three dimensional crack.

The asperity model

It has often been suggested in the literature that stress heterogeneity on the fault should be a source of high frequency radiation. We can analyze the effect of these variations in dynamic stress drop with our model. Stress heterogeneity produces variations of the stress intensity K_0, which in turn generate radiated waves. In this fashion stress variation will generate seismic radiation. Let us see this in more detail: K_0 depends on the dynamic stress drop via:

$$K_0(x) = \frac{1}{\pi} \int_{x-\beta t}^{x} \sigma_e(x_1) \frac{dx_1}{\sqrt{x - x_1}}$$

where as in (9) we assumed that the dynamic stress drop is independent of time. Clearly discontinuities in σ will be reflected in the variation of $K_0(x)$ although K_0 will be smoother because of the integration. Therefore, if there is a jump in stress at x_0, $K_0(x)$ will present an $(x - x_0)^{1/2}$ behaviour after the rupture front breaks through x_0. Assuming that the rupture velocity does not change at x_0, the velocity radiation calculated from (10) will present a $t^{1/2}$ type-wave-front. This is weaker wave than the step function fronts created by changes in rupture velocities. There is one case however in which the radiation due to stress heterogeneity will be as strong as that due to rupture velocity jumps. This occurs when the rupture front encounters a stress heterogeneity of the type:

$$\sigma_e(x) = K_a(x - x_0)^{-1/2}. \tag{18}$$

In this case the stress intensity changes at x_0 by:

$$K_0(x) = K_a H(x - x_0) \tag{19}$$

i.e., $K_0(x)$ jumps by the finite amount K_a.

This may appear as an extreme case of heterogeneity, yet it is very likely that it occurs on a fault plane subjected to successive events. Stress singularities of the type (18) are always associated with cracks, so that if the rupture is breaking into a previously unbroken patch it will almost certainly encounter a stress concentration of this kind. This is the case, for instance, in the asperity models studied by Rudnicki and Kanamori (1981) and Mc Garr (1981). In those models an unbroken patch has been left over from previous events on the fault. The stress concentration in the unbroken patch presents inverse square root singularities of the type (18) near the borders of the asperity. When the rupture front breaks through the asperity a jump in K_0 occurs and a strong jump in particule velocity is radiated. It is very likely, of course, that a jump in velocity will occur at the same time reinforcing even more the high frequency radiation.

In conclusion, there are two ways to produce jumps in the particle velocity radiation: in the first, the rupture front encounters a barrier where the strength or rupture resistance increases suddenly, the rupture velocity changes abruptly and a strong wave (step change in particle velocity) is generated. In the second case the rupture front encounters an asperity due to a previously unbroken ligament on the fault. Whether the rupture velocity changes or not, this generates a step in particle velocity. These two models are undistinguishable from the seismic radiation, unless we can detect the sign of the particle velocity jumps. Particle velocity jumps are associated with ω^{-1}-type high frequency asymptotes in the particle velocity spectrum. In terms of the displacement spectra these jumps create the usual ω^{-2} high frequency asymptotes. In acceleration both barriers and asperities of the type discussed above contribute to a flat high frequency spectrum of the type found by Hanks and Mc Guire (1981) in most accelerograms. Their results may be interpreted as a clear indication that high frequency waves are controlled by the presence of barriers and asperities on the fault plane.

CONCLUSIONS

We have solved analytically for the radiation of SH waves from an arbitrarily moving subsonic antiplane crack. The dynamic stress drop driving the slip and the motion of the rupture front was also allowed to be entirely arbitrary. With the further assumption that dynamic stress drop at a given point on the crack is independent of time once slip starts, i.e. that dynamic friction is constant and independent of slip, it was possible to give a very simple and compact form to the particle velocity field radiated by this crack model. The solution is exact and it involves no integrals ! As the rupture front moves, it continuously emits waves which are proportional to the local stress intensity factor and the instantaneous velocity of the rupture front. Abrupt changes in rupture velocity or stress intensity produce correspondingly sharp high frequency waves. Accelerograms are dominated by these impulsive waves. Barriers and asperities produce similar types of waves so that they would be difficult to distinguish solely from the high frequency near field waves that they generate.

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APPENDIX I

In order to reduce the integral (4) over the surface $S_1 + S_2 + S_3$ into the simpler integral (6) over S_1, we have to prove that the integrals over S_3 and S_2 are identically zero. The first integral over S_3 is obviously zero because it integrates the stress change σ_{xy} outside the crack, before the arrival of the S wave coming from the start of the rupture.

The proof that the integral over S_2 is also zero may be obtained if we convert from the x_1, t_1 coordinates into the ξ_1, η_1 system shown in figure 2. These are the characteristic coordinates introduced by Kostrov (1966):

$$\begin{aligned} \xi &= 2^{-1/2}(\beta t - x) \\ \eta &= 2^{-1/2}(\beta t + x) \end{aligned} \qquad \text{(I.1)}$$

with the Jacobian $J = \sqrt{2}/\beta$.

Then the integral over S_2 in (4) may be rewritten as :

$$u_y(\xi, z, \eta) = \frac{1}{\pi\mu}\int_0^{\xi} d\xi_1 \times$$

$$\times \int_{-\xi_1}^{\eta_m} d\eta_1 \frac{\sigma_{xz}(\xi_1, \eta_1)}{\sqrt{[(\xi - \xi_1)(\eta - \eta_1) - z^2/2]}} \qquad \text{(I.2)}$$

where

$$\eta_m = \eta - \frac{z^2}{2(\xi - \xi_1)}$$

is the equation for the hyperbola limiting the integration area S_2 from above (see fig. 2) : this is exactly the condition for R in (5) to be zero. Let us now split the integral over η_1 in two parts. The first is

$$I_1 = \int_{-\xi_1}^{\eta_R} d\eta_1 \frac{\sigma_e(\xi_1, \eta_1)}{\sqrt{(\xi - \xi_1)(\eta - \eta_1) - z^2/2}} \qquad \text{(I.3)}$$

where η_R is the intersection of the ξ_1 = constant line with the trajectory of the rupture front $\ell(t)$. In (I.3) we have replaced σ_{xy} with its value inside the crack, i.e. $\sigma_{xy} = \sigma_e$ as shown by the boundary condition (2a).

The second part of the integral over η_1 in (I.2) is

$$I_2 = \int_{\eta_R}^{\eta_m} d\eta_1 \frac{\sigma_{xy}(\xi_1, \eta_1)}{\sqrt{(\xi - \xi_1)(\eta - \eta_1) - z^2/2}} \qquad \text{(I.4)}$$

the stress $\sigma_{xy}(\xi_1, \eta_1)$ is the stress outside the crack which is given by (3). Rewriting that equation in the (ξ_1, η_1) coordinates we find

$$\sigma_{yz}(\xi, \eta) = -\frac{1}{\pi}\frac{1}{\sqrt{\eta - \eta_R}}\int_{-\xi_1}^{\eta_R} \frac{\sigma_e(\xi, u)\sqrt{\eta_R - u}}{\eta - u} du$$

and inserting this into (I.4) and changing the order of integration one obtains :

$$I_2 = -\frac{1}{\pi}\int_{-\xi_1}^{\eta_R} \sqrt{\eta_R - u}\ \sigma_e(\xi, u)\, du \times$$

$$\times \int_{\eta_R}^{\eta_m} \frac{d\eta_1}{(\eta_1 - u)\sqrt{\eta_1 - \eta_R}\sqrt{(\xi - \xi_1)(\eta - \eta_1) - z^2/2}}. \qquad \text{(I.5)}$$

The last integral may be evaluated by the residue theorem at the pole $\eta_1 = u < \eta_R$, this yields :

$$I_2 = -\int_{-\xi_1}^{\eta_R} \frac{\sigma_e(\xi, u)\, du}{\sqrt{(\xi - \xi_1)(\eta - u) - z^2/2}}. \qquad \text{(I.6)}$$

Comparing (I.6) with (I.3) one sees that $I_2 = -I_1$. This proves that the inner integral in (I.2) is identically zero and that the integral (4) over S_2 also vanishes. We thus obtain (6).

APPENDIX II

In the integral (6) we assume that

$$\sigma_e(x_1, t_1) = \sigma_e(x_1)$$

is independent of time t_1. In this case the time integral in (6) may be evaluated exactly. Let us rewrite (6) in the form

$$u_y(x, z, t) = \frac{\beta}{\pi\mu} \int^{\ell(\tau)} \sigma_e(x_1)\, dx_1 \times$$

$$\times \int_{t_0}^{t_M} \frac{dt_1}{\sqrt{\beta^2(t - t_1)^2 - (x - x_1)^2 - z^2}} \qquad (II.1)$$

where the limits of the time integral follow from the solution of (5) for $R = 0$, i.e.

$$t_M = t - \frac{\sqrt{(x - x_1)^2 + z^2}}{\beta} = t - \rho/\beta$$

this is the equation for the hyperbola in figure 2 limiting the integration area from above, and

$$t_0 = t - q/\beta$$

where

$$q = (x_1 - \ell(\tau)) - R^1$$

with R^1 given by (7). This is the equation of the characteristic through A in figure 2.
We rewrite (II.1) in the form

$$u_y(x, z, t) = \frac{1}{\pi\mu} \int^{\ell(\tau)} \sigma_e(x_1)\, dx_1 \int_\rho^q \frac{du}{\sqrt{u^2 - \rho^2}}. \qquad (II.2)$$

The integral (II.2) may be simplified even further if we notice that the last integral is $\cosh^{-1}(q/\rho)$. But since we are interested in particle velocities we differentiate (II.2) with respect to time, this yields

$$v_y(x, z, t) = \frac{1}{\pi\mu} \int^{\ell(\tau)} \frac{\sigma_e(x_1)}{\sqrt{q^2 - \rho^2}} \frac{dq}{dt} dx_1 \qquad (II.3)$$

which is the same as (8) in the main text. the time derivative of q is

$$\frac{dq}{dt} = \frac{dq}{d\ell}\frac{d\ell}{dt}$$

$$\frac{dq}{dt} = \left(1 - \frac{x - \ell}{R^1}\right) \frac{\dot{\ell}}{1 - \dot{\ell}/\beta(x - \ell)/R^1}$$

where $d\ell/dt$ was calculated differentiating (7) with respect to time.
After rearranging we obtain :

$$\frac{dq}{dt} = \dot{q} = \frac{R^1 - (x - \ell)}{R^1 - \dot{\ell}/\beta(x - \ell)}\, \dot{\ell}$$

as in the text.
Finally notice that

$$q^2 - \rho^2 = 2(\ell - x_1)\left[R^1 - (x - \ell)\right]$$

which when inserted into (8) leads to (9).

REFERENCES

Achenbach J. D., Harris J. A., 1978. Ray methods for elastodynamic radiation from a slip zone of arbitrary shape, *J. Geophys. Res.*, **83**, 2283-2291.

Aki K., 1967. Scaling law of seismic spectrum, *J. Geophys. Res.*, **72**, 1217-1231.

Aki K., Richards P. G., 1980. Quantitative Seismology, W. F. Freeman, San Francisco, CA.

Andrews D. J., 1981. A stochastic fault model, 2, Time-dependent case, *J. Geophys. Res.*, **86**, 10821-10834.

Das S., Aki K., 1977. Fault plane with barriers : a versatile earthquake model, *J. Geophys. Res.*, **82**, 5658-5670.

Eshelby J. D., 1969. The elastic field of a crack extending non-uniformly under general antiplane loading, *J. Mech. Phys. Solids*, **17**, 177-179.

Freund L. B., 1972. Wave motion in an elastic solid due to a non uniformly moving line load, *Quart. App. Math.*, **30**, 271-281.

Hanks T. C., Mc Guire R. K., 1981. The character of high frequency strong ground motion, *Bull. Seismol. Soc. Am.*, **71**, 2071-2995.

Haskell N. A., 1966. Total energy and energy spectral density of elastic wave radiation from propagating fault. Part II. A statistical source model, *Bull. Seism. Soc. Am.* **56**, 125-140.

Ida Y., 1973. Stress concentration and unsteady propagation of longitudinal shear cracks, *J. Geophys. Res.*, **78**, 3418-3429.

Kostrov B. V., 1966. Unsteady propagation of longitudinal shear cracks, *J. Appl. Math. Mech.*, **30**, 1241-1248.

Madariaga R., 1978. The dynamic field of Haskell's rectangular dislocation fault model, *Bull. Seism. Soc. Am.*, **68**, 869-887.

Madariaga R., 1977. High-frequency radiation from crack (stress drop) models of earthquake faulting, *Geophys. J. R. astr. Soc.*, **51**, 625-651.

Mc Garr A., 1981. Analysis of peak ground motion in terms of model of inhomogeneous faulting, *J. Geophys. Res.*, **86**, 3901-3912.

Rudnicki J., Kanamori H., 1981. Effects of fault interaction on moment, stress drop, and strain energy release, *J. Geophys. Res.*, **86**, 1785-1793.

Slepjan L. I., 1981. Theory of Cracks, Sudostrojenj, Leningrad.