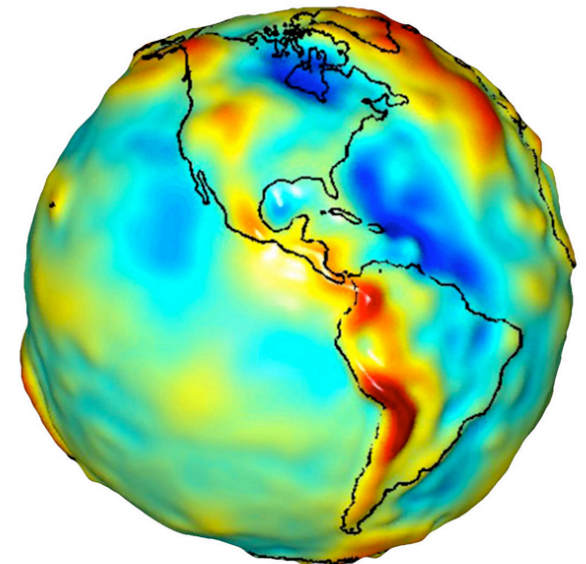# Elements of Geodesy Part 1

Shape and Figure of the Earth
Potential and Geoid
Terrestrial Coordinate Systems

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Geoid of Western Hemisphere. Image from University of Texas Center for Space Research and NASA.

The shape of the Earth: a sphere

Pythagore

- An oyster… (the Babylonians before 3000 B.C.)
- Pythagore (582-507 B.C.):
  - Speculative considerations based on aesthetics
  - Sphere and number 10 considered perfect => perfect world would consist of 10 planetary bodies of spherical form
  - First to suggest the earth rotated in a circular motion around the sun
- Aristotle (4th century B.C.):
  - All heavy bodies tend to fall towards the center and seek the lowest place in this process thus they are eventually pressed into spherical ball
  - Observations: Ships at sea, superstructure is last seen going away from land
  - Changes in observable stars going north/south
  - Earth cast round shadows during lunar eclipses
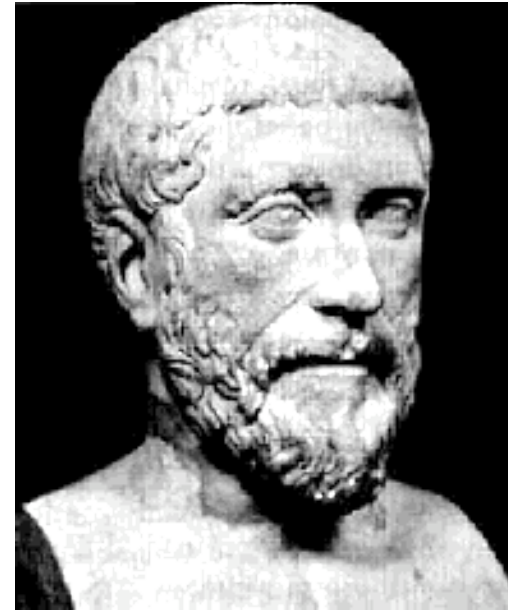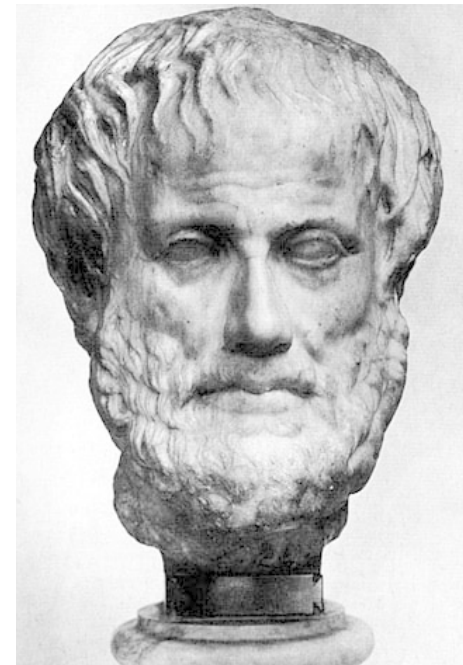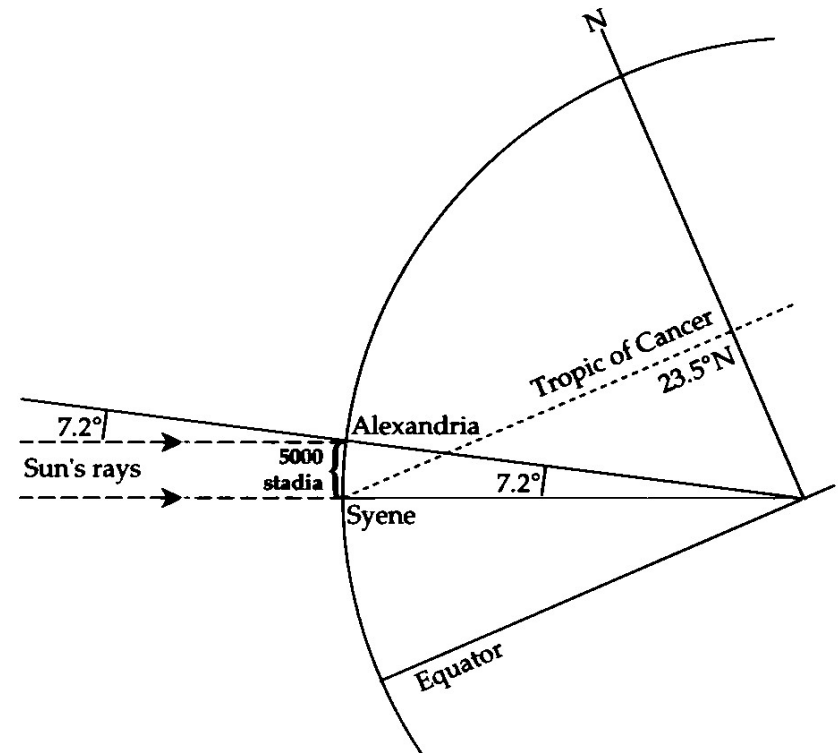

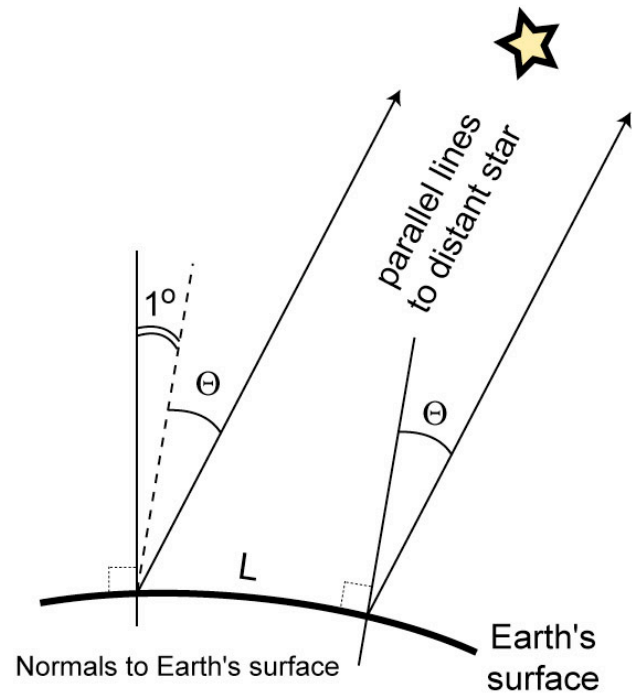Aristotle

The shape of the Earth: a sphere

Erathostene

- Erathostene (275-195 B.C.), head librarian in Alexandria, Egypt:
  - 2 wells in Assouan (=Syene) and Alexandria
  - Sun rays vertical at Assouan at noon on mid-summer day, but shadow on well side in Alexandria
  - Knowing the Assouan-Alexandria distance = 10 camel days = 5000 stadia
  - 5000 stadia x 185 = 925 000 m: Earth's radius R = 925 000/(7.2 x p/180) = 7 364 650 m
  - Earth circumference = 2 x p x R = 46 250 000 m, 40 030 000 m in reality => 15% error
  - Assouan and Alexandria not exactly on the same meridian
- Spherical Earth models often used for short range navigation (VOR-DME), for global distance approximations, in seismology, plate kinematics, etc...: ~14 km difference for polar radius

Measuring latitudes and longitudes

parallel lines to distant star

1°

Θ

Θ

L

Normals to Earth's surface

Earth's surface

L = length of 1 degree of meridian

- Latitudes:
  - Zenith angle of the Sun at Noon (accounting for declination from a table)
  - Corollary: length of a degree from zenith angle to distant star

- Longitudes:
  - More difficult than latitude
  - Use Earth's rotation (360 degrees per day): Local time - time at reference site = longitude…
  - Local time: max. sun elevation at noon
 - Time at reference site: need accurate clock.
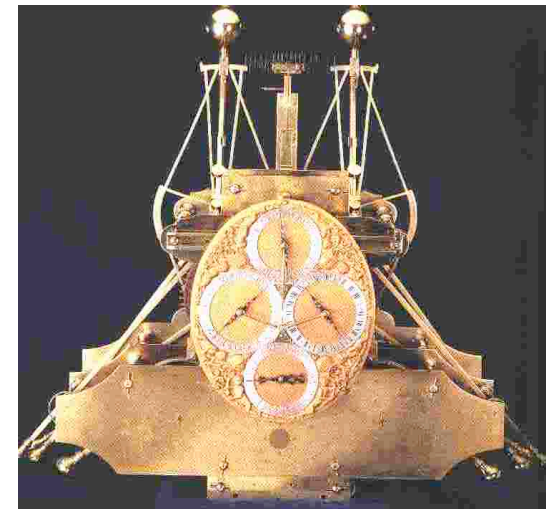
The H1 chronograph (Harrison, 1737)

The shape of the Earth: NOT a sphere…

- Jean Picard, 1669, measured length of meridian arc from Malvoisine (N. of Paris) to Amiens using triangulation and astronomic latitude at ends
- Continued north to Dunkerque and south to Collioure by Philippe de Lahire and Jacques and Dominique Cassini between 1683 and 1716
- Result:
  - Picard = 57,060 toises
  - Cassini N = 56,960 toises
  - Cassini S = 57,097 toises
  - Interpretation:
    - 1 degree meridional arc decreases northward
    - Sphere tangent to the pole has a smaller radius than at the equator, therefore Earth is "egg-shaped" prolate spheroid
    - Measurement errors?
- First evidence Earth not sphere, but…
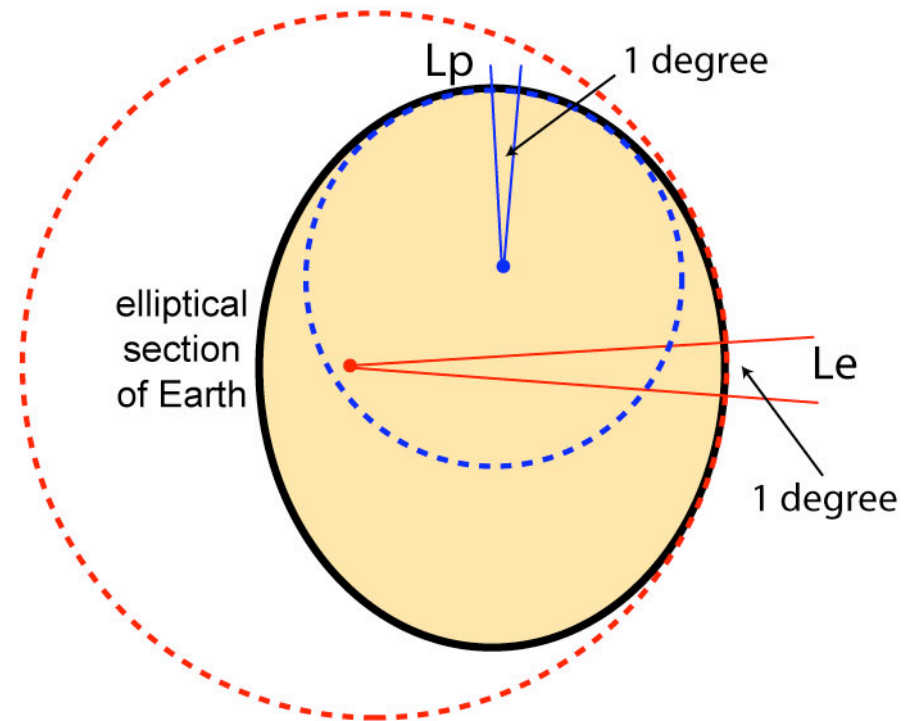
The shape of the Earth: an ellipsoid

- Huygens, then Newton:
  - Theoretical considerations: Earth rotation + fluid => should be elliptical and flattened at the poles
  - Newton computes flattening = 1/230 (1/298 in reality)
  - French Academy of Sciences orders a test…

- Two expeditions for measuring the length of one degree of meridian:
  - Clairaut and Maupertuis, 1736, Laponia
  - Bouguer and Condamine, 1743, Peru
  - Result: 1 degree longer in Laponia than in Peru
  - Sphere tangent to the pole has a larger radius than at the equator, therefore the poles are flattened
  - First definition of the meter: *"10 million[th] part of a quarter of the circumference of an Earth's meridian"*

elliptical section of Earth
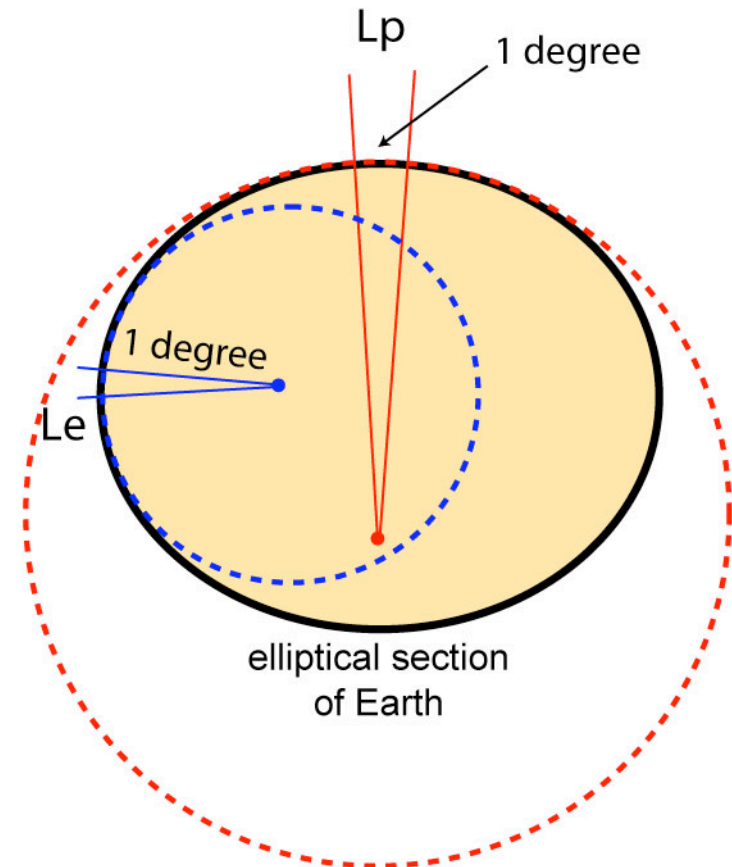
Parameterization of the ellipsoidal Earth

The shape of the Earth is mathematically represented as an ellipsoid defined by:

- Semi-major axis = equatorial radius = **a**
- Semi-minor axis = polar radius = **b**
- Flattening (the relationship between equatorial and polar radius):

$$f = \frac{a-b}{a}$$

- Eccentricity:

$$e = \frac{\sqrt{a^2 - b^2}}{a} \Rightarrow e^2 = 2f - f^2$$

$R - b \approx 14.2\ km$ sphere

$a - R \approx 7.1\ km$ R b a
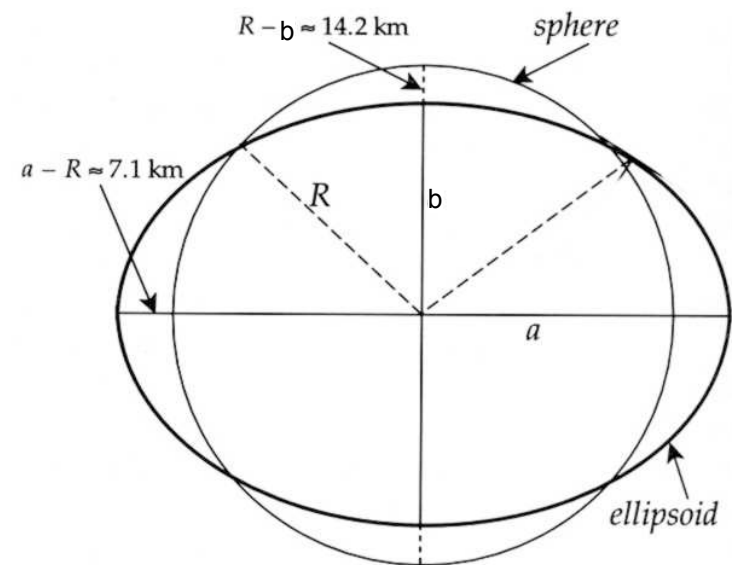
ellipsoid

$a\ =\ 6378.136\ km$
$b\ =\ 6356.751\ km$
$R\ =\ 6371.000\ km$

Comparison between the WGS-84 ellipsoid and a sphere of identical volume

Coordinates on the ellipsoidal Earth

- Given that the shape of the Earth is close to an ellipsoid, it is convenient to define a position by its latitude, longitude, and height = ellipsoidal (or geodetic) coordinates:
  - The Prime Meridian is the origin for longitudes. The Equator is the origin for latitudes.
  - Geodetic latitude ϕ = angle from the equatorial plane to the vertical direction of a line normal to the reference ellipsoid.
  - Geodetic longitude λ = angle between a reference plane and a plane passing through the point, both planes being perpendicular to the equatorial plane.
  - Geodetic height h = distance from the reference ellipsoid to the point in a direction normal to the ellipsoid.
- Other quantities:
  - Geocentric latitude ϕ'
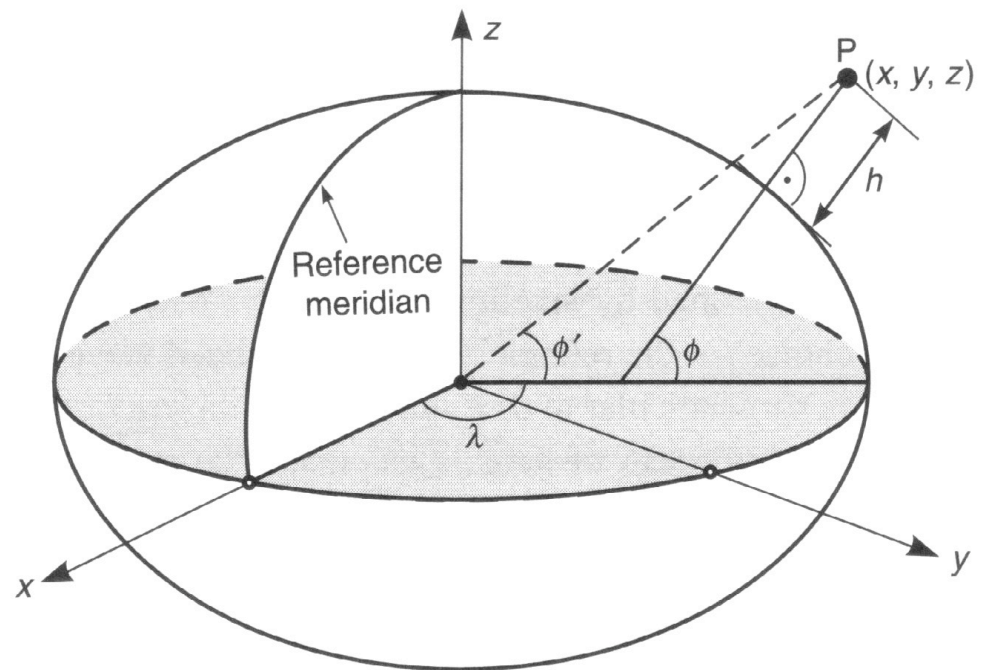
Earth Centered, Earth Fixed (ECEF)

- Simplest coordinate system is ECEF:
  - Origin = center of mass of the Earth.
  - Three right-handed orthogonal axis X, Y, Z.
  - Units are meters.
  - The Z axis coincides with the Earth's rotation axis.
  - The (X,Y) plane coincides with the equatorial plane.
  - The (X,Z) plane contains the Earth's rotation axis and the prime meridian.
- Sometimes associated with the notion of "spherical coordinates" = coordinates on sphere of mean Earth radius:
  - Spherical latitude
  - Spherical longitude

Earth Centered, Earth Fixed X, Y, Z

P H Dana 8/17/94
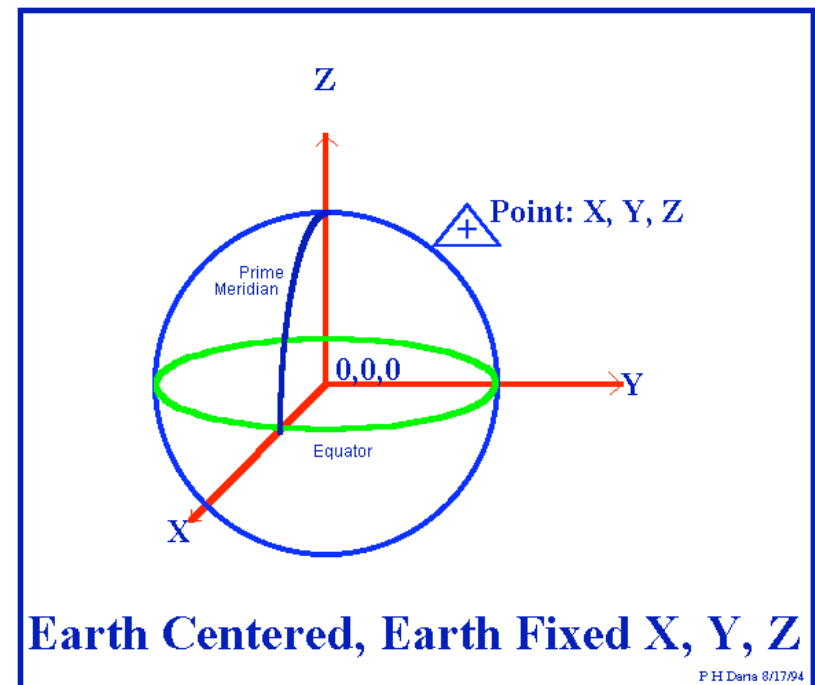
Coordinate systems

Conversion between ECEF and ellipsoidal coordinates can be made using the following formulas:

Coordinate Conversion
Geodetic Latitude, Longitude, and Height to ECEF, X, Y, Z

$X = (N + h)\cos\phi\cos\lambda$

$Y = (N + h)\cos\phi\sin\lambda$

$Z = [N(1 - e^2) + h]\sin\phi$

where:

ϕ, λ, h = geodetic latitude, longitude, and height above ellipsoid

X, Y, Z = Earth Centered Earth Fixed Cartesian Coordinates

and:

$N(\phi) = a / \sqrt{1 - e^2 \sin^2\phi}$ = radius of curvature in prime vertical

a = semi-major earth axis (ellipsoid equatorial radius)

b = semi-minor earth axis (ellipsoid polar radius)

$f = \dfrac{a - b}{a}$ = flattening

$e^2 = 2f - f^2$ = eccentricity squared

Peter H. Dana 8/3/96

Coordinate Conversion: Cartesian (ECEF X, Y, Z) and Geodetic (Latitude, Longitude, and Height)
Direct Solution for Latitude, Longitude, and Height from X, Y, Z

This conversion is not exact and provides centimeter accuracy for heights < 1,000 km (See Bowring, B. 1976. Transformation from spatial to geographical coordinates. Survey Review, XXIII: pg. 323-327)

$\phi = \text{atan}\left(\dfrac{Z + e'^2 b \sin^3\theta}{p - e^2 a \cos^3\theta}\right)$

$\lambda = \text{atan2}(Y, X)$

$h = \dfrac{p}{\cos(\phi)} - N(\phi)$

where:

ϕ, λ, h = geodetic latitude, longitude, and height above ellipsoid

X, Y, Z = Earth Centered Earth Fixed Cartesian coordinates

and:

$p = \sqrt{X^2 + Y^2}$ $\theta = \text{atan}\left(\dfrac{Za}{pb}\right)$ $e'^2 = \dfrac{a^2 - b^2}{b^2}$

$N(\phi) = a / \sqrt{1 - e^2 \sin^2\phi}$ = radius of curvature in prime vertical

a = semi-major earth axis (ellipsoid equatorial radius)

b = semi-minor earth axis (ellipsoid polar radius)

$f = \dfrac{a - b}{a}$ = flattening

$e^2 = 2f - f^2$ = eccentricity squared

Topocentric system (or datum)

- Coordinates can also be referring to a local **topocentric datum**.
  - Origin = any point on the surface of the Earth.
  - 3 orthogonal axis: u (for "up") is vertical and points upwards, n (for "north") is in the local horizontal plane and points to the geographic north, e (for "east") is in the local horizontal plane and points to the geographic east.
  - Left-handed.
  - Units = meters.
- The conversion is a combination of:
  - 1 translation, so that system origins coincide
  - 1 reflection (left- to right-handed system)
  - 2 rotations (to align the ECEF axis with the NEU axis)

$$\begin{bmatrix} n \\ e \\ u \end{bmatrix} = \begin{bmatrix} -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{bmatrix} \begin{bmatrix} X - Xo \\ Y - Yo \\ Z - Zo \end{bmatrix}$$

where *[X,Y,Z]* is the vector to be transformed (in meters), *[Xo,Yo,Zo]* the position of the origin of the NEU system, and λ and ϕ the longitude and latitude of that origin, respectively.
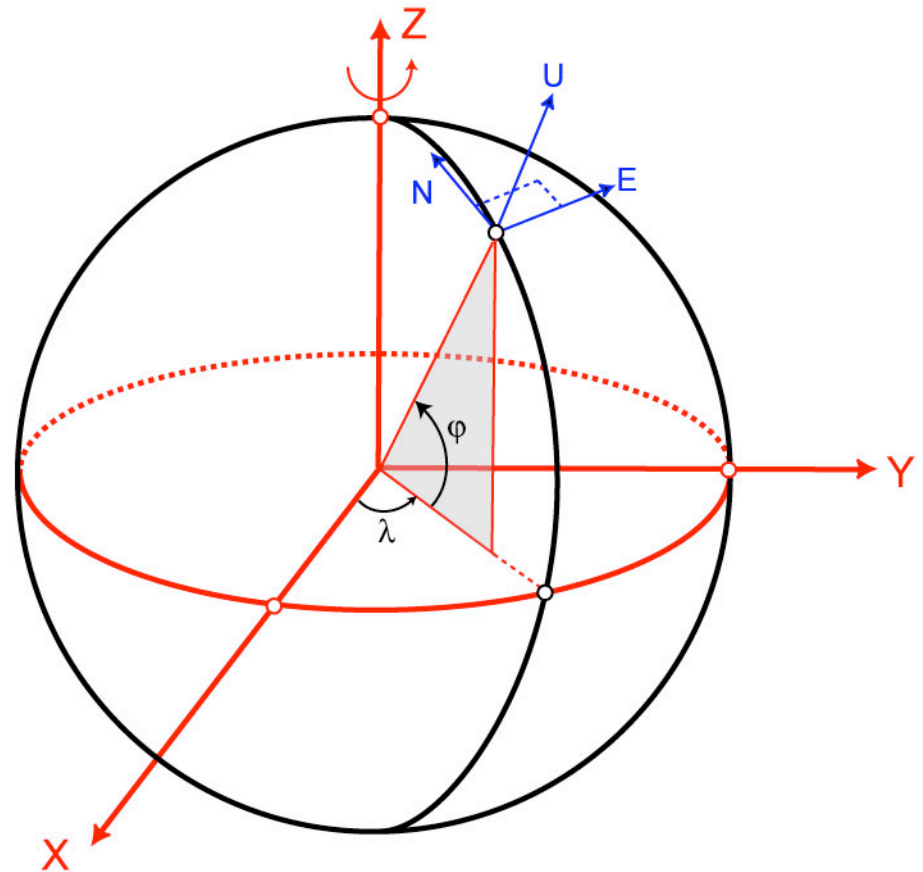
The Figure of the Earth

- **Geoid** = surface of constant equipotential energy
- Recall that gravitational acceleration of unit mass due to the attraction of mass M is:

$$\vec{g} = G\frac{M}{r^2}\vec{r}$$

- The particle is attracted by M and moves toward it by a small quantity dr -- this displacement is the result of work W exerted by the gravitational field generated by M:

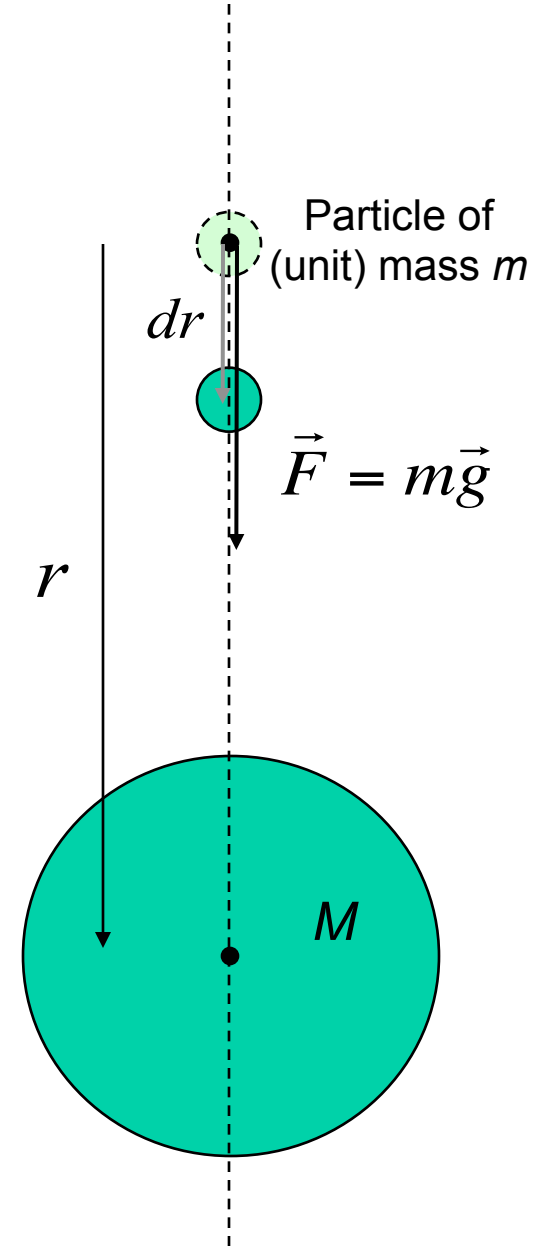$$W = F.dr = mg.dr \Rightarrow W = G\frac{M}{r^2}dr$$

- The potential U of mass M is the amount of work necessary to bring the particle from infinity to a given distance r:

$$U = \int_\infty^r G\frac{M}{r^2}dr = GM\int_\infty^r \frac{1}{r^2}dr = GM\left[\frac{1}{\infty} - \frac{1}{r}\right]$$

$$\Rightarrow U = -\frac{GM}{r}$$

and: $g = -\nabla U$

- At distance r, the gravitational potential generated by M is U

Particle of (unit) mass m

dr

$\vec{F} = m\vec{g}$

r

M

Gravitational potential

- Spherically Symmetric, Non-rotating Earth: $U = \dfrac{GM}{r}$

  => geoid = sphere of radius R

- Rotating spherically symmetric Earth: one must account for frame rotation w.r.t. inertial frame, which results in a "centrifugal force"

$$U = \frac{GM}{r} + \frac{1}{3}\omega^2 r^2 - \frac{1}{3}\omega^2 r^2 P_2 \qquad \text{with} \qquad P_2 = \frac{1}{2}\left(3\cos^2\theta - 1\right)$$

- Rotating ellipsoidal Earth: additional term due to departure from sphere (caused by P_2 term above)

Angular-dependent terms

$$U = \frac{GM}{r} + \boxed{\frac{1}{3}\omega^2 r^2 - \frac{1}{3}\omega^2 r^2 P_2} - \boxed{\frac{GM}{r^3}a_e^2 J_2 P_2}$$

Rotation terms   oblateness term

- θ = colatitude
- ω = angular rotation rate (rad/sec)
- a_e = equatorial radius
- J_2 = quantifies oblateness

The reference ellipsoid

- It turns out that the shape of the equipotential surface as defined above (*i.e.,* to degree 2) is an ellipsoid = **reference ellipsoid.**

- It is a "mathematical surface".

- Four parameters needed to define that surface: *a, GM, J$_2$, ω*

- Adopted by an international body (under the International Association of Geodesy), several versions through time.

- Current system = **Geodetic Reference System 1980** (GRS80)

- Based on the theory of the geocentric equipotental ellipsoid, with:
  - *a* = 6,378,137 m
  - *GM* = 398,600.5 x 10^9 m^{3}s^{-2}
  - *J$_2$* = 1,082.63 x 10^{-6}
 - *ω* = 7,292,115 x 10^{-11} rad s^{-1}

The "actual" geoid

- The Geoid = the particular equipotential surface that coincides with mean sea level
- Over the oceans, the geoid is (would be) the ocean surface (if there were no currents, waves, etc)
- Over the continents, the geoid is not the topographic surface (its location can be calculated from gravity measurements)
- Actual shape of geoid is complex because:
  - Heterogeneous mass distribution in the Earth
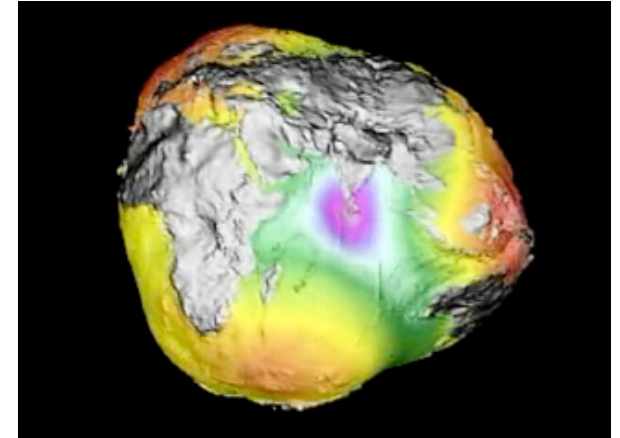 - Time-depence of mass distribution

topography

ocean surface
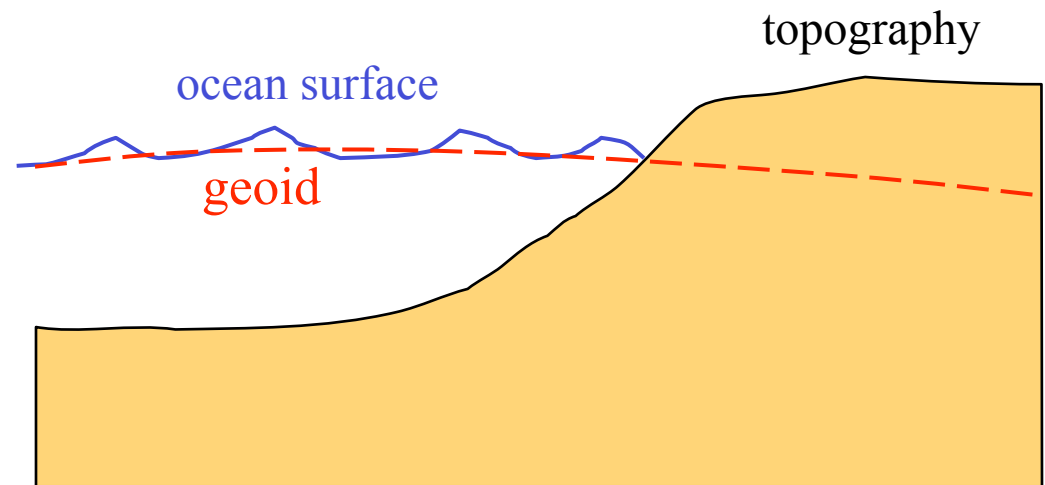
geoid

The "actual" geoid

- Actual shape of equipotential is not an ellipsoid, primarily because of heterogeneous distribution of mass in the Earth:
    - Excess of mass $\Rightarrow$ geoid (ocean surface) goes up (geoid = eqp surface $\Rightarrow$ must remain perpendicular to the gravity field direction!)
    - Deficit of mass $\Rightarrow$ geoid (ocean surface) goes down
- Results in "geoid anomalies" = differences, in meters, between the geoid and the reference ellipsoid   (= geoid "height"):
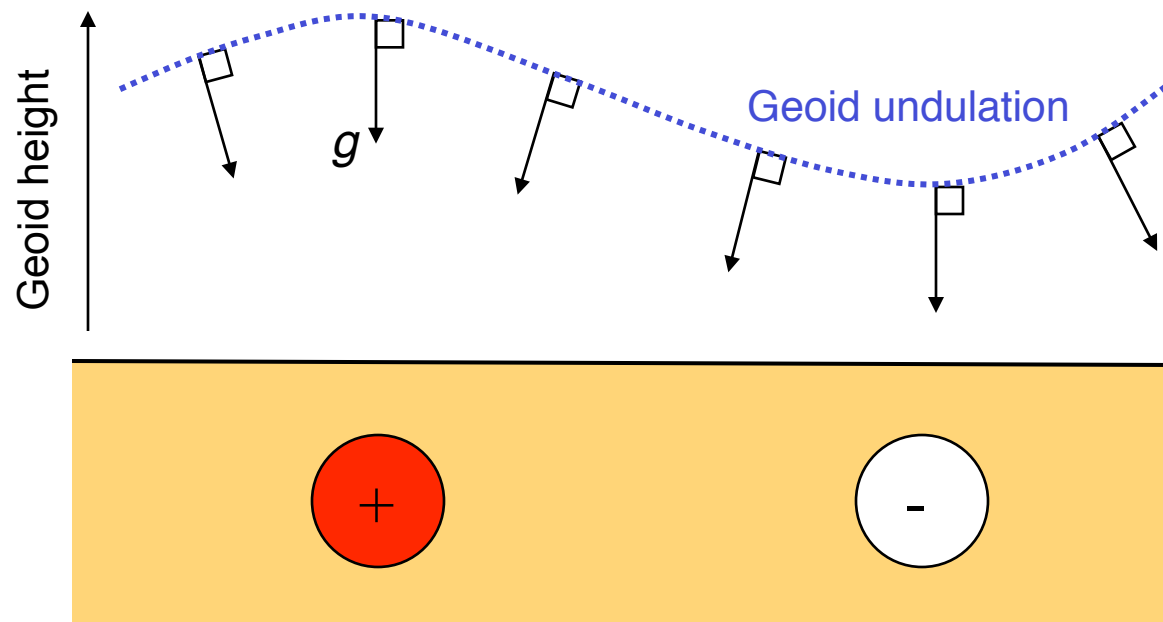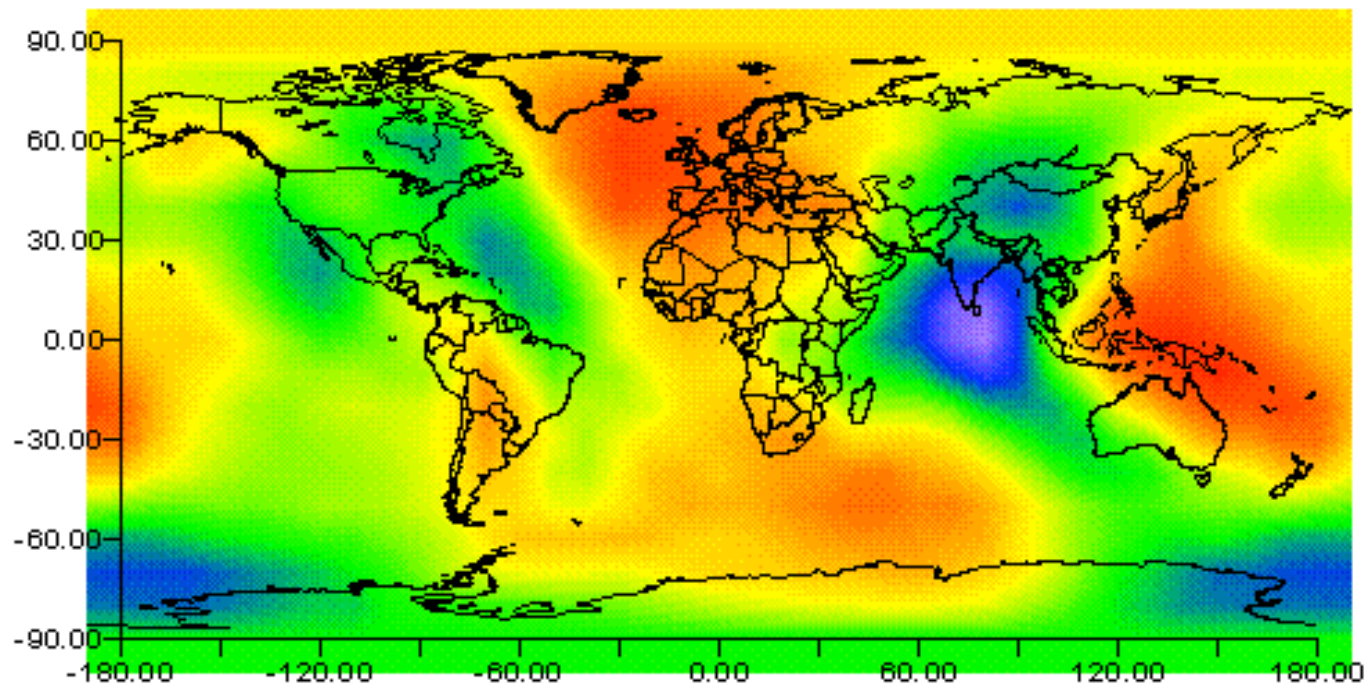
The Geoid

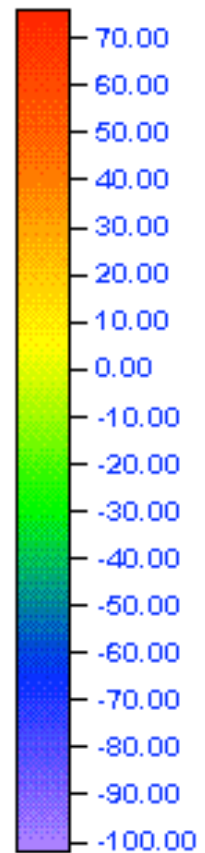- Geoid = the "figure" of the Earth (="potatoid"…?)
- Geoid height varies (globally) within +- 100 m

WGS-84 Geoid Height

From DMA 10 by 10 Degree Geoid Height Grid

The geoid of the Conterminous U.S.

United States Hybrid Geoid for 2003 (GEOID03)

MIN = −51.009 m   MAX = 3.428 m   AVE = −29.488 m   STD = 10.106 m
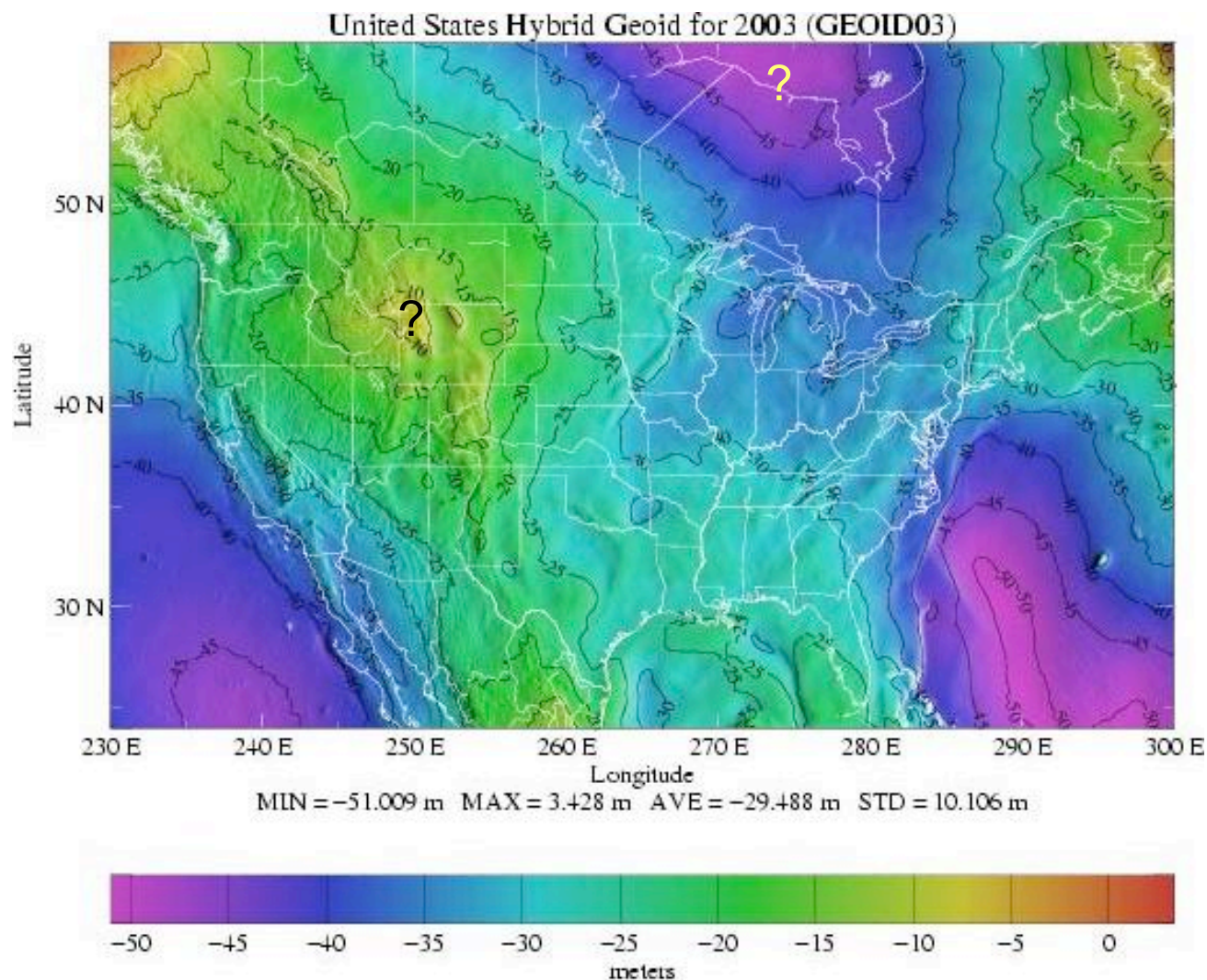
Geoid heights vary from 50 m (Atlantic) to 5 m (Rocky Mountains)
www.ngs.noaa.gov/GEOID/

Coordinate systems - Height

- Conventionally heights are measured above an equipotential surface corresponding to mean sea level (MSL) = the geoid

- Topographic maps, markers, etc.: height above or below mean sea level = orthometric height, H.

- Height above or below a reference ellipsoid (e.g. from GPS reading) = ellipsoidal height, h.

- Difference between h and H = geoid height N.

- Angle between directions of the ellipsoidal normal and plumb line at given point = deflection of the vertical

- Transformation from ellipsoidal height to orthometric height requires to know the local geoid height.

H	Orthometric height
h	Ellipsoidal height
N	Geoidal height
ε	Deflection of the vertical
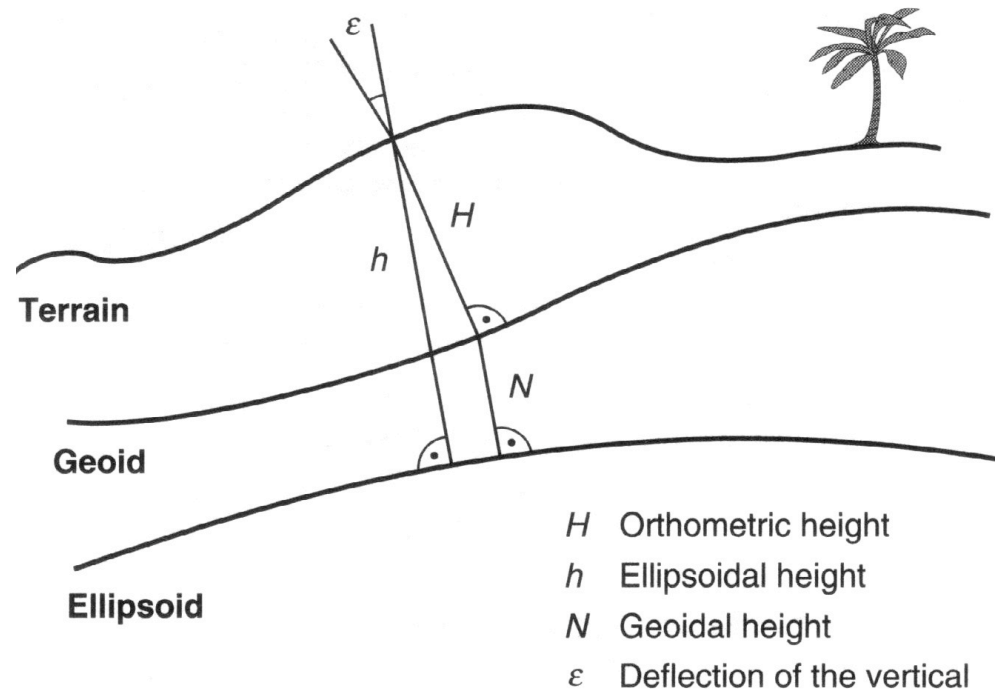
$$=> h = H + N$$

The World Geodetic System

- Defined by U.S. DoD for satellite positioning.
- Defining parameters:
  - Shape: same as GRS80 with very small difference in *f* => semi-minor axis differ by 0.0001 m.
  - Origin: center of mass of the whole Earth (solid + oceans + atmosphere)
  - Orientation (*i.e.*, X, Y, Z-axis):
    - Early versions:
      - Z-axis = direction of Conventional Terrestrial Pole at epoch 1984.0 as defined by BIH
      - X-axis = coincident with zero-meridian at epoch 1984.0 as defined by BIH
      - Y-axis: completes the right-handed ECEF orthogonal geocentric coordinate system.
      - Defined using satellite Doppler coordinates: 1-2 m accuracy
    - Revised in 1994 to WGS84(G730) to include:
      - GPS-derived coordinates at 10 DoD tracking stations + 22 IGS => few cm accuracy.
      - Some IERS products used (ITRF) => new origin and orientation!
- Revised to WGS84(G873) in 1996 then WGS84(G1150) in 2001 to conform to ITRF origin and orientation.

Reference vs. regional ellipsoids

How could this happen? (well, it did not…)

- Global Reference System (GRS) and World Geodetic System (WGS):
  - Based on the ellipsoid that best fits the geoid.
  - Geocentric
- Regional ellipsoids:
  - Best-fit to a local geoid, e.g. for a particular country
  - Not necessarily geocentric
- Latitude, longitude, and height of a given point are different if different ellipsoids are used.
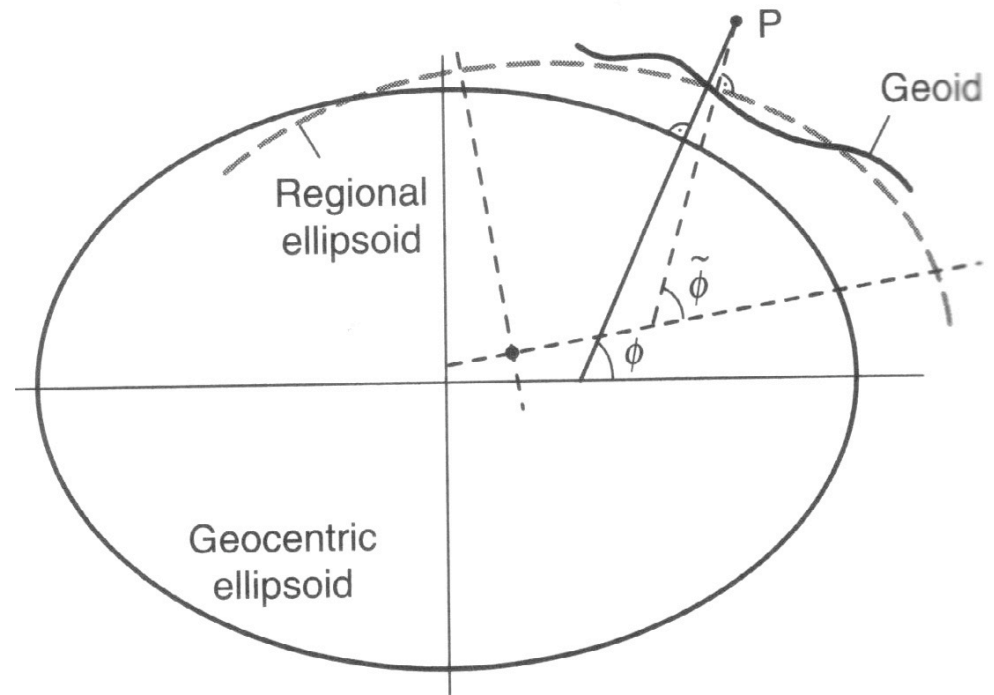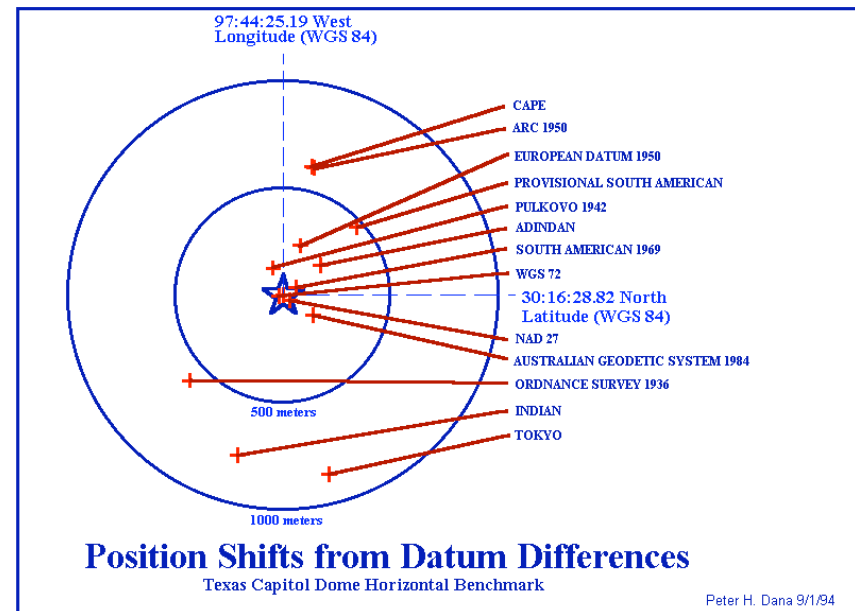
Datum

- Datum =
    1. The size and shape of the Earth, approximated as an ellipsoid: semi-major axis a and flattening f.
    2. The translation of its origin with respect to the Earth's center of mass (t_X, t_Y, t_Z)

- Many different datums have been defined and are in use.

- Conversion between datums:
    - Translation
    - 3 rotations (usually small angles)
    - 1 scale

- Great care must be used when manipulating coordinates to ensure that they are associated with a well-defined datum.

Name	Date	a (m)	b (m)	Use
Everest	1830	6377276	6356079	India, Burma, Sri Lanka
Bessel	1841	6377397	6356079	Central Europe, Chile, Indonesia
Airy	1849	6377563	6356257	Great brittain
Clarke	1866	6378206	6356584	North America, Philippines
Clarke	1880	6378249	6356515	France, Africa (parts)
Helmert	1907	6378200	6256818	Africa (parts)
International (or Hayford)	1924	6378388	6356912	World
Krasovsky	1940	6378245	6356863	Russia, Eastern Europe
GRS80	1980	6378137	6356752	North America
WGS84	1984	6378137	6356752	World (GPS measurements)

http://www.kartografie.nl

97:44:25.19 West
Longitude (WGS 84)

CAPE
ARC 1950
EUROPEAN DATUM 1950
PROVISIONAL SOUTH AMERICAN
PULKOVO 1942
ADINDAN
SOUTH AMERICAN 1969
WGS 72

30:16:28.82 North
Latitude (WGS 84)

NAD 27
AUSTRALIAN GEODETIC SYSTEM 1984
ORDNANCE SURVEY 1936
INDIAN
TOKYO

500 meters

1000 meters

Position Shifts from Datum Differences
Texas Capitol Dome Horizontal Benchmark

Peter H. Dana 9/1/94

Vertical datum

Fragment of the Dutch topo map showing the border of Belgium and the Netherlands. The Mean Sea Level of Belgium differ -2.34m from the MSL of The Netherlands. As a result , contour lines are abruptly ending at the border (http://www.kartografie.nl).
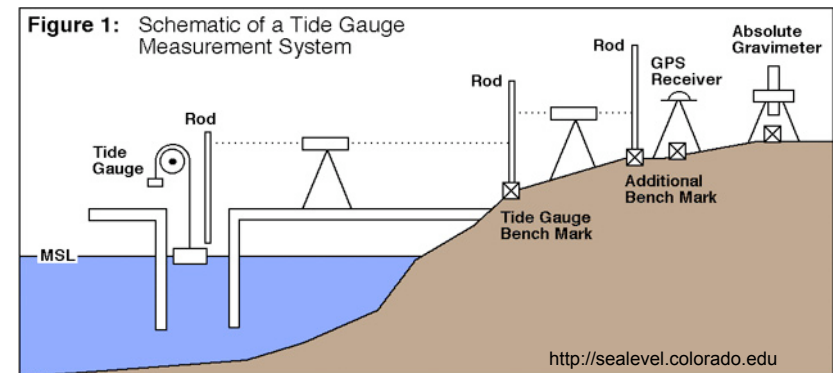
- Latitude, longitude expressed in a given ellipsoidal reference system = the theoretical geoid (from potential theory, see previous slides)
- But in practice heights are expressed with respect to Mean Sea Level = the actual geoid…
- MSL reference:
  - Different for each country
  - Established using tide-gauge measurements if access to ocean.
  - Usually some average ot tide heights.
  - Carried over using leveling.
- North American Vertical Datum = NAVD 88:
  - Primary tidal benchmark in Rimouski, Québec, Canada.
  - Minimum-constraint adjustment of the Canadian-Mexican-U.S. leveling observations.
- Problem: MSL is not the same equipotential at all tidal gauges… (why?)

Tide gauges => relative sea level, i.e. w.r.t. Earth crust, but crust may move vertically w.r.t. geocenter… => GPS measurements at tide gauges can be used to measure "absolute" sea level

Satellite altimetry

- Satellite carries radar => ocean – satellite range
- Ground tracking system => ellipsoid – satellite distance
- Difference = ocean – ellipsoid distance = dynamic topography
- Precision:
  - SEASAT (1979) = 10 cm
  - TOPEX-POSEIDON (1992) = 5 cm
  - JASON (launched in 2002) < 5 cm
- Dynamic topography contains:
  - Oceanographic effects: waves, currents, tides
  - Gravimetric effects = the geoid
- Satellite altimetry can be used to:
  - Obtain the geoid (see http://topex.ucsd.edu/marine_grav/mar_grav.html)
 - Monitor global sea level
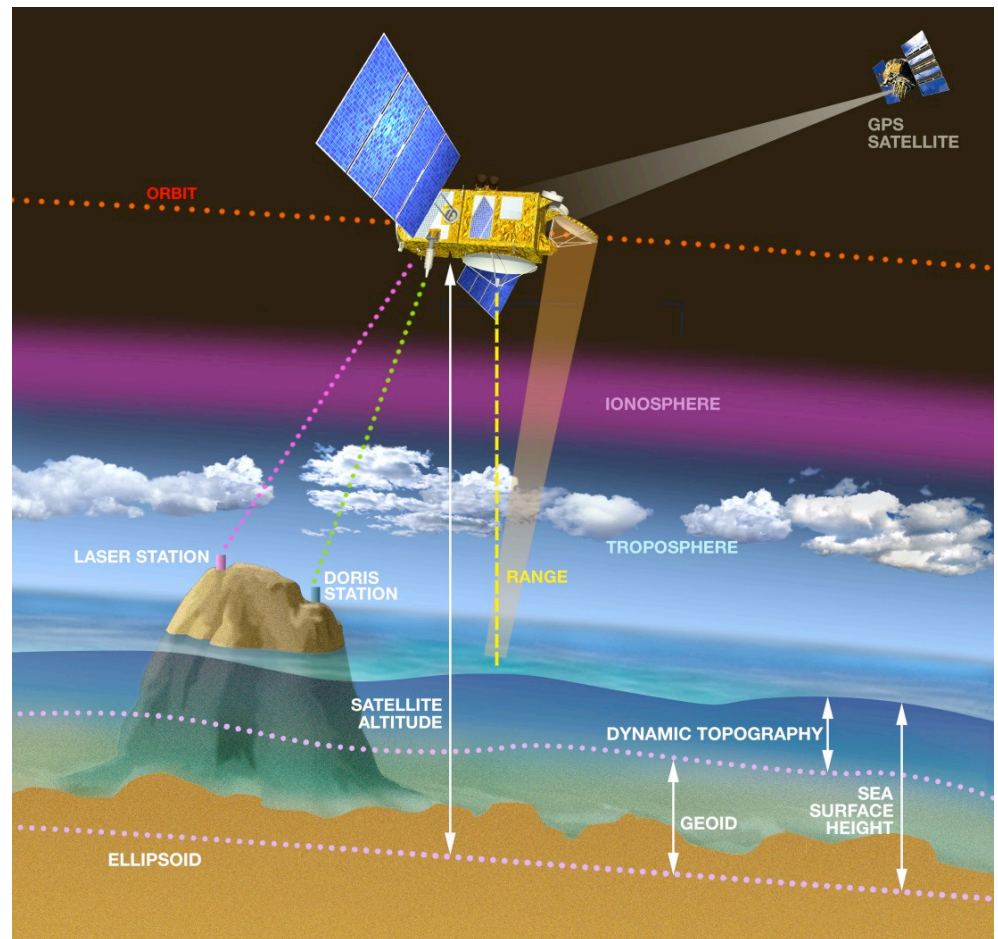
Satellite altimetry

- Global sea level is rising at ~3 mm/yr

- Sea-level trends vary geographically

- Results from 2 processes:
    - Steric component: volume increase with temperature, between 0.4 and 1.9 mm/yr
    - Eustatic component: Melting of ice sheets
    - Relative contribution debated.

- Melting of ice sheets and sea-level rise affect Earth rotation…
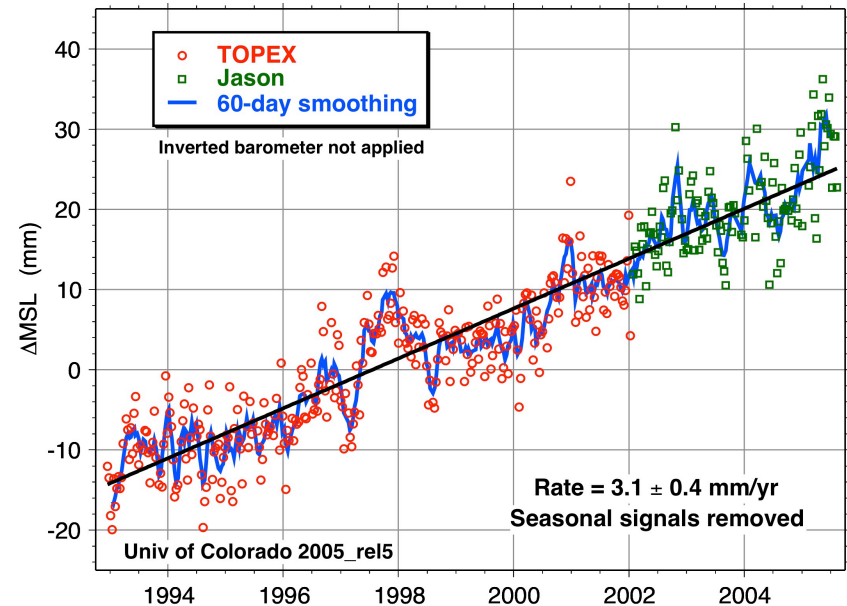
Trends in sea level, 1993-2005 (http://sealevel.colorado.edu)
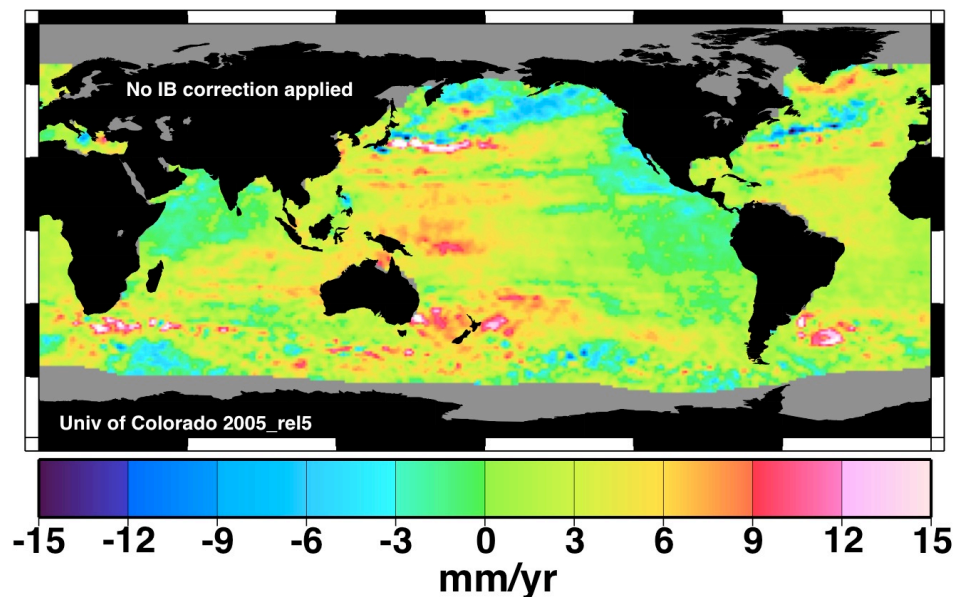
Datums and frames: some issues

- Is a datum, or coordinate system sufficient, in practice?
  - No, need to be able to access it physically = a frame.
  - Need for reference stations whose coordinates are well-defined in a given coordinate system
  - Those stations constitute the frame.
  - Example of WGS-84: how do GPS users tie their measurements to WGS84?
- Most datums are accessible through station coordinates from old surveys: GPS can be more precise…
- Problem:
  - Datums are typically static: they actually move and deform…
  - Exacerbated when using global or continent-wide datum…
- Datums and frames cannot be perfect since they rely on imperfect (but constantly improved) models and measurements.
- Consistent datums across borders:
  - Horizontal => use large-scale datums, continent-wide or even global
  - Transform between local datums, (and keep using old trig. network)
  - Vertical: can be tricky since reference is usually **actual** geoid and not ellipsoid…
- Changing datum:
  - Costs $$$.
  - Has legal implications.