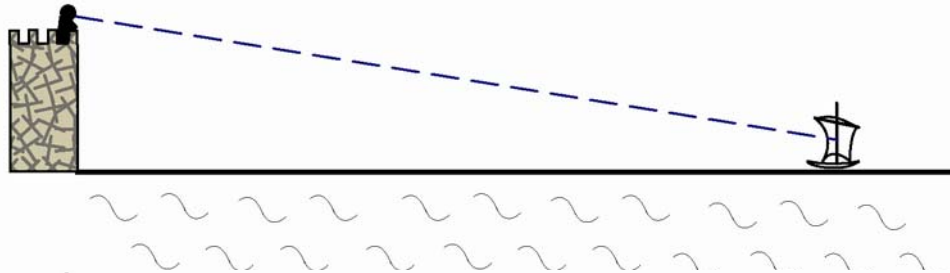


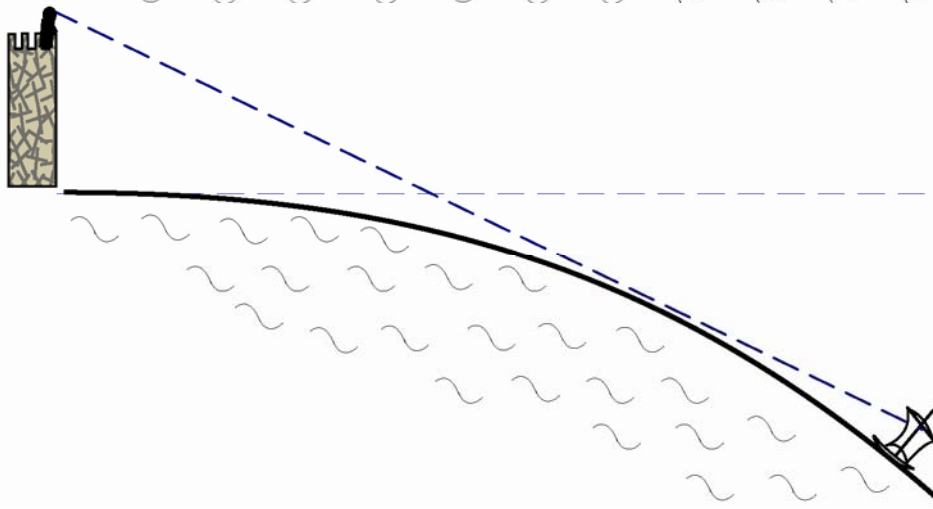
Ancient times Geodesy (6 century bc)

- Geodesy is a very old science. It comes from the first question mankind ask themselves : **what is the shape and the size of the earth ?**



If the Earth were flat,
then one could see
very far

=> no horizon

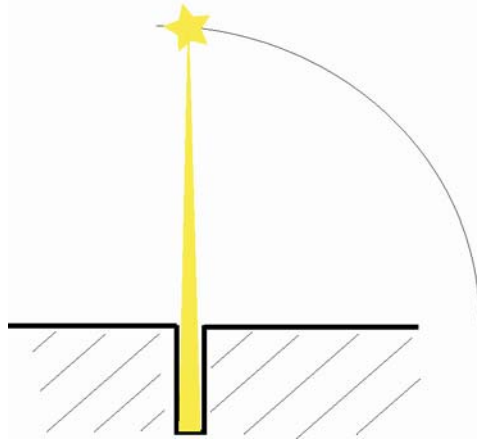


Because there is an
horizon (i.e. objects
disappear below the
horizon)

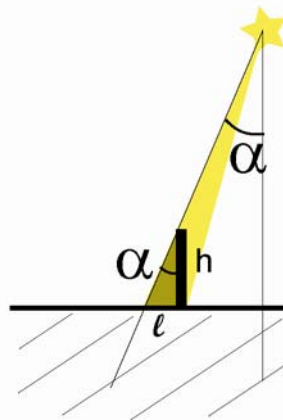
⇒ **Earth is spherical**

Ancient times Geodesy (Eratosthene, 300 bc)

$$\text{Size of the Earth : } \text{circ} = 360^\circ / \alpha * d_{12} = 40000 \text{ km}$$

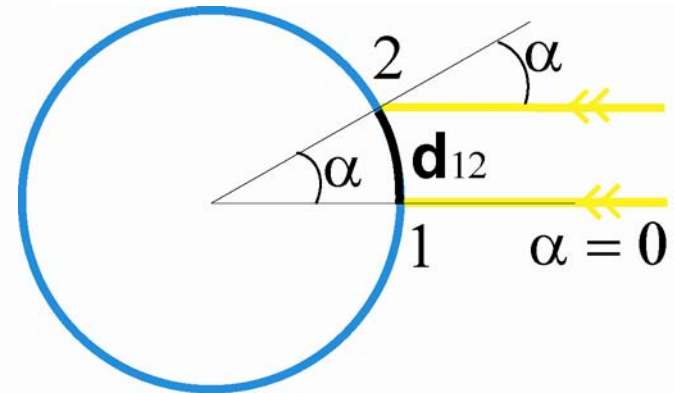


At one place on Earth, the Sun is vertical (lights the bottom of a well) only once a year



At the same time, at a different place, the Sun is not vertical

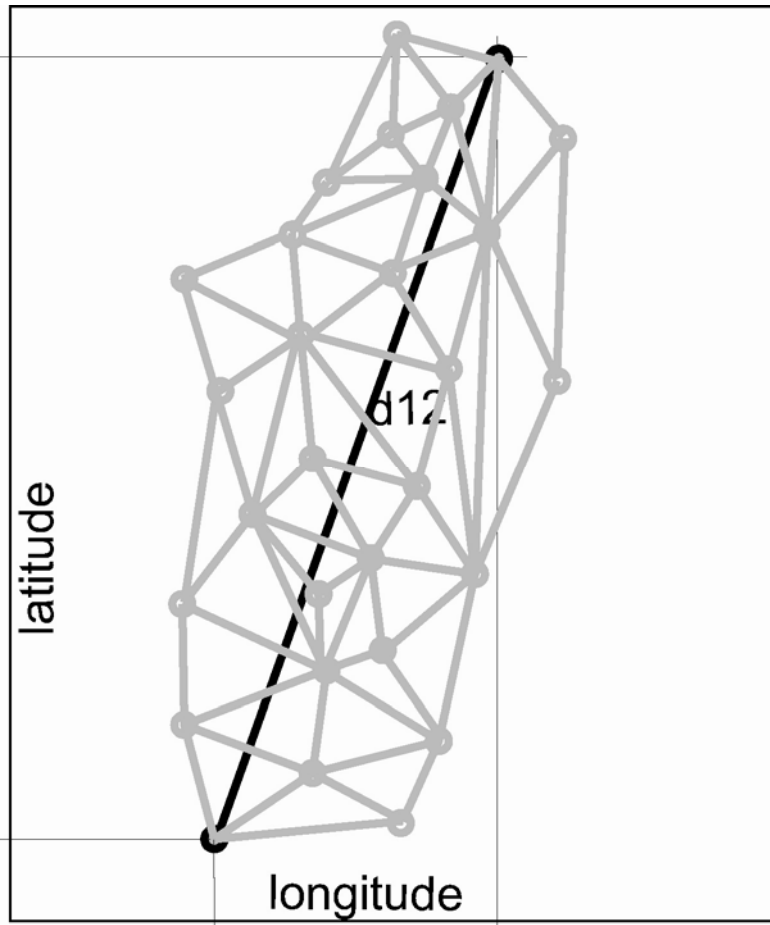
The angle can be measured from the length of the shadow of a vertical pole



The angle α of the sun light direction depends on the **local vertical** direction

=> Depends on the **latitude** of the site

«Modern» Geodesy (17th century)



A correction has to be made if distance is **not aligned** with longitude

d_{12} can be computed from the **sum** (oriented) of many smaller distances

Measuring many (if not all) **distances** and **angles** within a network of points give the more accurate solution for d_{12}

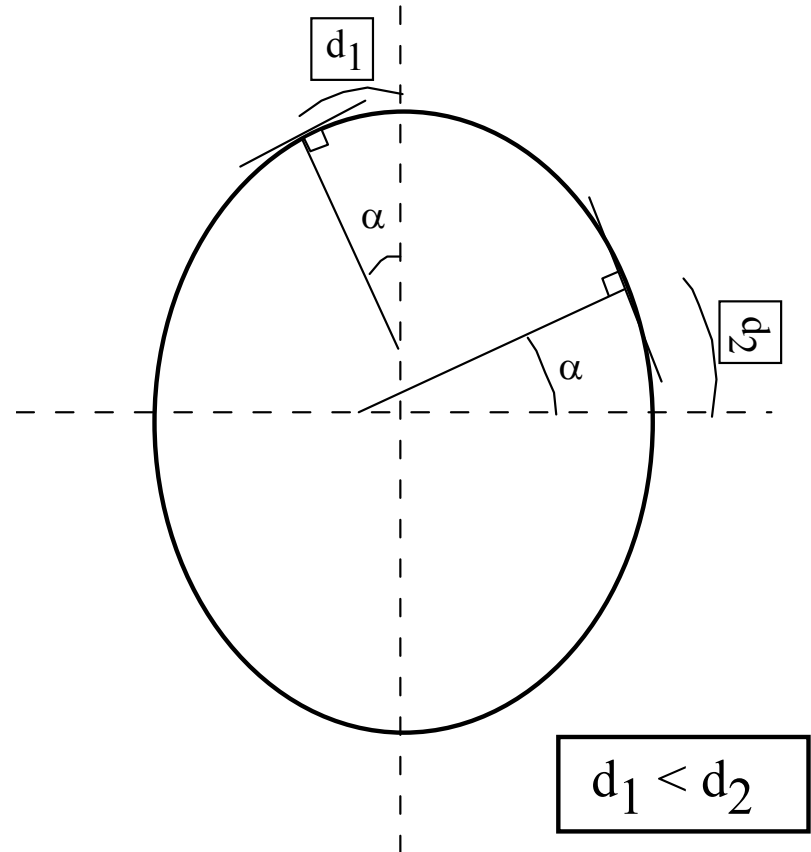
The shape of the Earth (18th century)

Making those measurements, different people find different values for the length of an arc of 1° at different places in Europe

- Snellius (1617) : 104 km
- Norwood (1635) : 109 km
- Riccioli (1661) : 119 km

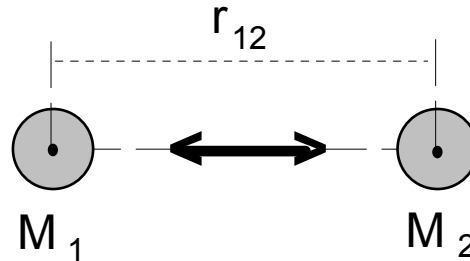
In France, **Picard** finds :

- **108 km** in the **north** of France
- **110 km** in the **south** of France



Force gravitationnelle

1/ attraction réciproque de deux masses



Les deux masses M_1 et M_2 s'attirent mutuellement avec une force F_{12} telle que :

$$\vec{F}_{12} = -G \frac{M_1 M_2}{r^2} \vec{u}_{12}$$

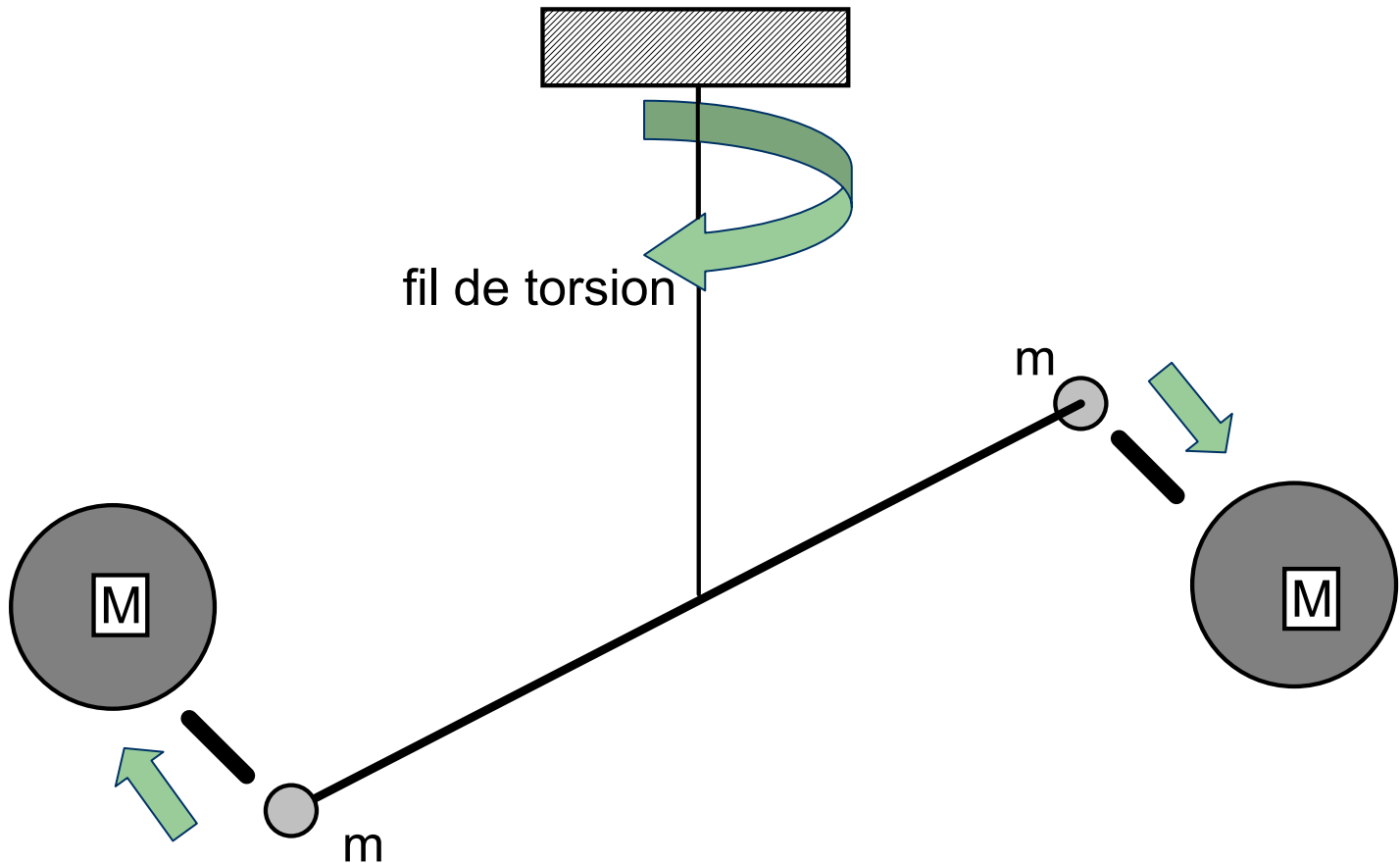
où G est la constante de gravitation universelle : $G = 6.674 \cdot 10^{-11}$ S.I.

$$[G] = [\text{force}] \cdot [\text{longueur}]^2 / [\text{masse}]^2$$

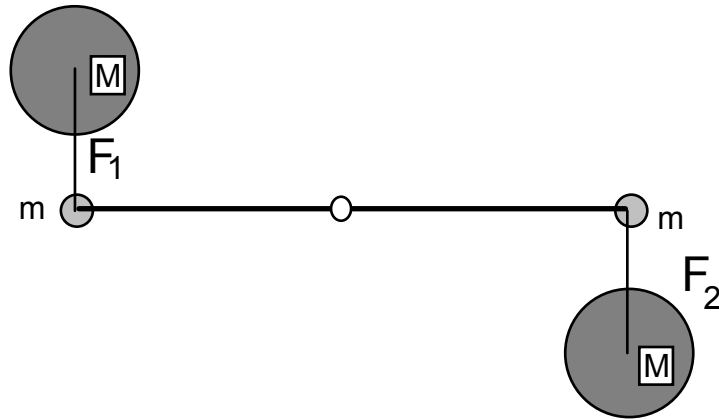
$$= \text{kg} \cdot \text{m} \cdot \text{s}^{-2} \cdot \text{m}^2 \cdot \text{kg}^{-2}$$

$$= \text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$$

Determination de G : experience de Cavendish



Determination de G: experience de Cavendish



chacune des grosses masses exerce une attraction sur la petite masse la plus proche (on néglige l'effet sur la petite masse la plus lointaine), et provoque une rotation du pendule.

les forces gravitationnelles F_1 et F_2 valent : $\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F} = \mathbf{GMm}/r^2$ (où r est la distance entre la grosse et la petite masse.)

le couple gravitationnel vaut donc : $\mathbf{L.F} = \mathbf{GMm.L}/r^2$

et le couple de torsion du pendule vaut : $\mathbf{C.\theta}$

à l'équilibre, les deux couples s'annulent : $\mathbf{GMm.L}/r^2 = \mathbf{C.\theta}$ ce qui donne $\mathbf{G}$.

Determination de G: experience de Cavendish

Une fois G connue, on en déduit la masse et la densité de la Terre. Sachant que la force d'attraction à la surface vaut environ $F = 10 \text{ m/s}^2$ (valeur donnée par la mesure de la vitesse de la chute des corps), on peut poser :

$$F = \frac{GM_{\text{Terre}}}{R_{\text{Terre}}^2}$$

avec $R_{\text{Terre}} = 6400 \text{ km}$,

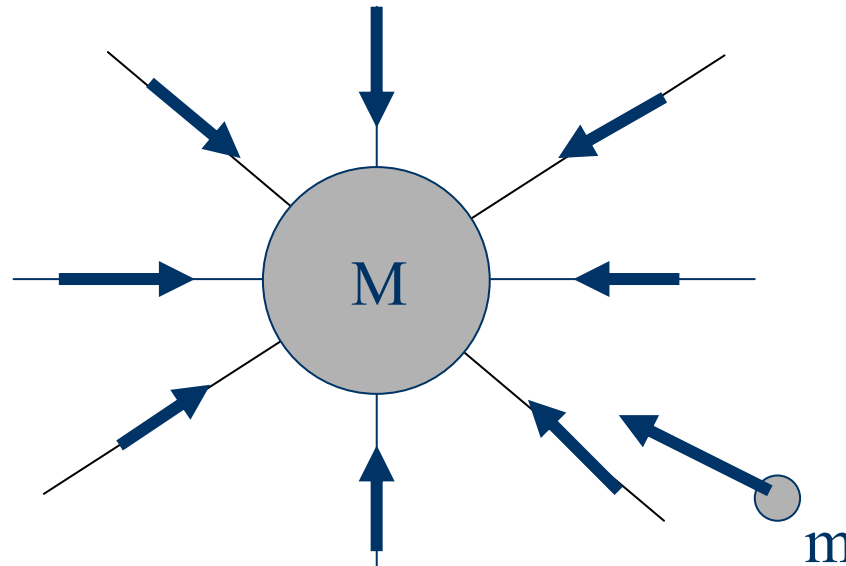
on trouve :

$$M_{\text{Terre}} = 6 \cdot 10^{24} \text{ kg}$$

et :

$$\rho_{\text{Terre}} = 5500 \text{ kg/m}^3$$

Champ de gravité créé par une masse



Une masse **M** placée dans l'espace va créer un « champ de gravité » qui va occuper tout l'espace

$$\vec{g}(x, y, z) = \vec{g}(r, \theta, \varphi) \quad (= GM/r^2 \text{ dans ce cas particulier})$$

Dans le champ de gravité (**g**) ainsi créé, toute particule de matière de masse **m**, placée dans ce champ, subira une force de gravitation **F = m g**

Potentiel de gravité

le rotationnel du champ de gravité est nul :

$$\vec{\nabla} \wedge \vec{u} = \begin{bmatrix} \left[\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \cdot u_\varphi) - \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (u_\theta) \right] \vec{e}_r \\ \left[\frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (u_r) - \frac{1}{r} \frac{\partial}{\partial r} (r \cdot u_\varphi) \right] \vec{e}_\theta \\ \left[\frac{1}{r} \frac{\partial}{\partial r} (r \cdot u_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta} (u_r) \right] \vec{e}_\varphi \end{bmatrix}$$

avec

$$\vec{g} = \begin{bmatrix} \frac{GM}{r^2} \cdot \vec{e}_r \\ 0 \cdot \vec{e}_\theta \\ 0 \cdot \vec{e}_\varphi \end{bmatrix} \quad (\text{fonction de } r \text{ uniquement})$$

on obtient aisément

$$\vec{\nabla} \wedge \vec{g} = \vec{0}$$

Potentiel de gravité

Or, le rotationnel du gradient d'un champ est toujours nul (relation fondamentale des opérateurs n° 4).

$$\text{rot}(\text{grad}V) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} = \begin{bmatrix} \frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y} = 0 \\ \frac{\partial^2 V}{\partial z \partial x} - \frac{\partial^2 V}{\partial x \partial z} = 0 \\ \frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x} = 0 \end{bmatrix}$$

donc il existe un champ scalaire V tel que : $\vec{g} = -\text{grad}(V)$

Alors, le rotationnel de g sera alors automatiquement nul: par construction. On dit que g dérive d'un potentiel, et que V est le potentiel de gravité.

Dans le cas simple précédent où g vaut GM/r^2 , il est évident que le potentiel V vaut : GM/r

Potentiel de gravité

D'autre part, la divergence du champ de gravité est nulle :

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \cdot u_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (u_\varphi)$$

Et là encore, seule la composante de $\mathbf{g}$ selon r est non nulle, et vaut $\mathbf{g}_r = -GM/r^2$.

On a donc :

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{-GM}{r^2} \right) + 0 + 0 = 0$$

On a donc bien : $\text{div}(\mathbf{g}) = 0 \Leftrightarrow \text{div}(\text{grad}(V)) = 0 \Leftrightarrow \text{Laplacien}(V) = 0$

Le potentiel de gravité est donc un champ scalaire à Laplacien nul.

On pourra donc l'exprimer sur la base des harmoniques sphériques !!!

Développement en harmoniques sphériques

Le potentiel de gravité terrestre observé peut donc s'écrire sur la base des harmoniques sphériques :

$$Y_l^m(r, \theta, \varphi) = (a_l^m r^l + b_l^m r^{-(l+1)}) P_l^m(\cos \theta) e^{im\varphi}$$

à l'évidence, le potentiel ne peut pas devenir infini quand r devient très grand (et qu'on s'éloigne de la source). Le seul polynôme en r qui ait un sens physique est donc celui qui décroît avec r , c'est à dire le polynôme en $1/r^{(l+1)}$. Le potentiel s'écrit donc :

$$\begin{aligned} V(r, \theta, \varphi) &= \sum_{l=0}^{+\infty} \sum_{m=-l}^l V_{l,m} \cdot Y_l^m(r, \theta, \varphi) \\ &= \sum_{l=0}^{+\infty} \left(\frac{R}{r}\right)^{l+1} \sum_{m=-l}^l V'_{l,m} \cdot Y_l^m(\theta, \varphi) \\ &= \frac{GM}{r} \left[1 + \sum_{l=1}^{+\infty} \left(\frac{R}{r}\right)^l \sum_{m=-l}^l V''_{l,m} \cdot Y_l^m(\theta, \varphi) \right] \end{aligned}$$

$$V(r, \theta, \varphi) = \frac{GM}{r} \left[1 + \sum_{l=1}^{+\infty} \left(\frac{R}{r} \right)^l \sum_{m=-l}^l V_{l,m}'' \cdot Y_l^m(\theta, \varphi) \right]$$

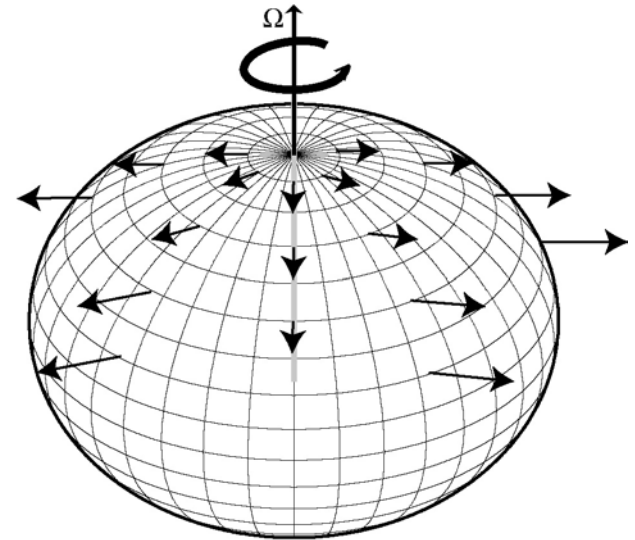
- Au premier ordre ($l=0$), on retrouve bien le potentiel sphérique $V = GM/r$. Les termes suivants représentent donc des *perturbations* par rapport à ce potentiel simple qui serait celui d'une planète à symétrie sphérique.
- La dépendance radiale du potentiel de gravité est donnée par le terme en $(R/r)^{l+1}$ qui indique que plus on s'éloigne de la source du champ, plus les ondulations à courtes longueur d'onde sont lissées. On retrouve donc un potentiel sphérique si on s'éloigne suffisamment de la planète ($r \gg R$).
- Le choix comme référentiel de description d'un repère dont l'origine est le centre de masse de la planète, permet d'annuler les termes de degré $l=1$, qui correspondent à un décalage du centre du potentiel (centre de masse) suivant chacun des trois axes.

Potentiel hydrostatique

Newton et **Huygens** introduisent les notions de gravités et d'accélération centrifuge :

$$g = GM/R^2 \quad \text{and} \quad F = R \Omega^2$$

=> une sphère molle en rotation doit s'aplatir aux pôles



la force centrifuge due à la rotation Ω , vaut sur l'équateur : $F_c = R\omega^2$

la force de gravité vaut : $F_g = GM/R^2$

Le rapport des deux forces vaut donc : $\alpha = F_c/F_g = R^3\omega^2/GM$

L'aplatissement théorique, c'est à dire la différence des rayons équatorial et polaire rapportée au rayon moyen, est de l'ordre de α .

La controverse sur la forme de la Terre

Newton, Huygens et Maupertuis avaient raison.

Picard avait tort.

Les mesures de la longueur d'un arc méridien de 1° :

- Bouguer et La Condamine et Jussieu en Equateur (publiées en 1744 par Bouguer) **56753 toises à 0°**
- Maupertuis et Clairaut en Laponie (publiées en 1737) **57438 toises à 66°N**
- (1 toise = 1,949 m)
- $\Rightarrow$ aplatissement 1/179 !!! (en fait 1/298)
- $\Rightarrow$ Attention aux incertitudes

Développement Y_{lm} du Potentiel hydrostatique

Il est facile de réaliser que l'hypothèse d'un aplatissement aux pôles conduit à proposer pour la Terre un potentiel de forme simple et symétrique:

En effet, la symétrie des forces agissantes (symétrie cylindrique) implique que la variation du potentiel ne peut être que :

- latitudinale (seule la latitude joue) $\Rightarrow m = 0$
- symétrique par rapport à l'équateur $\Rightarrow l$ pair

On obtient donc un potentiel dit « hydrostatique » :

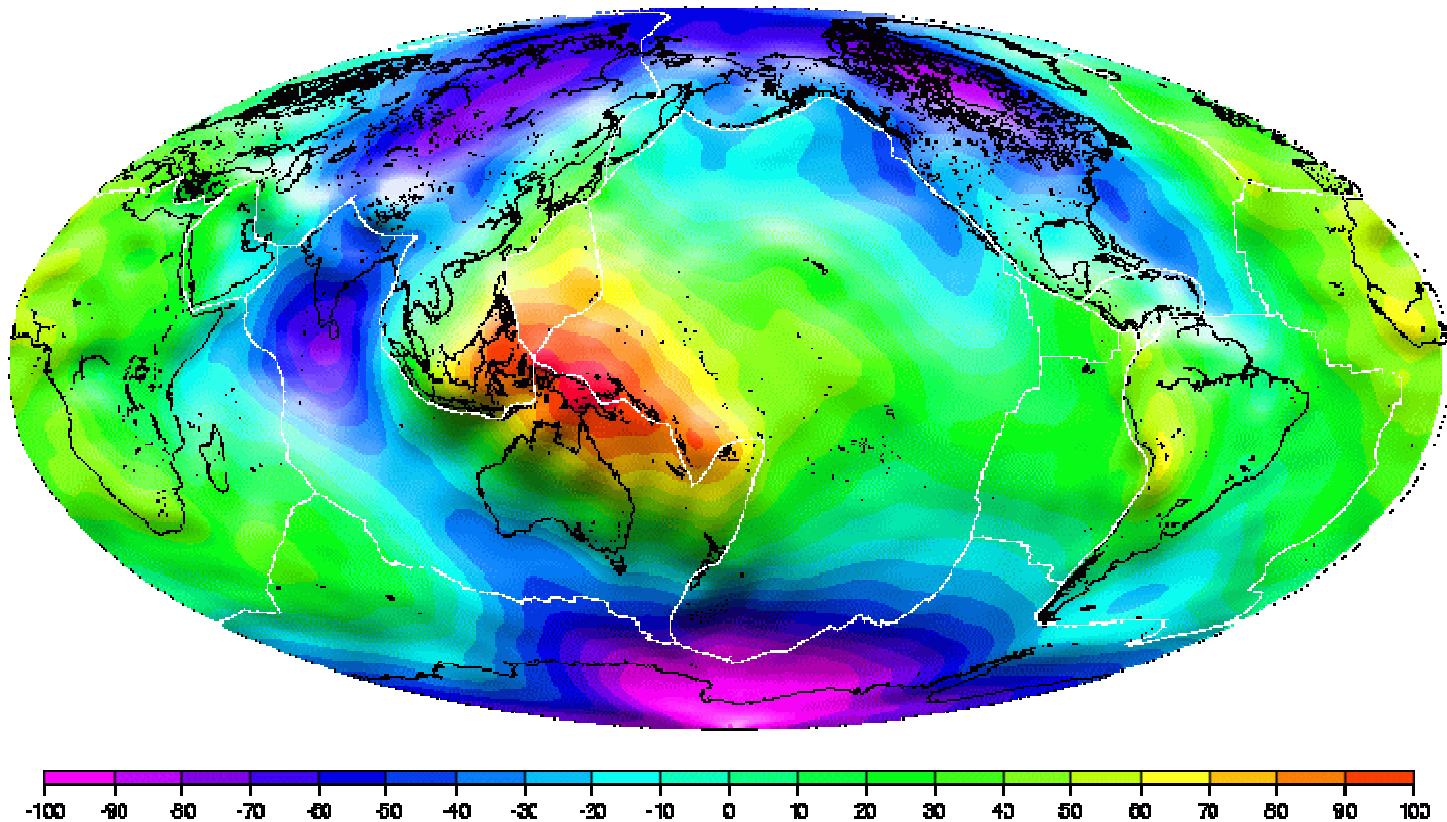
$$V_H(r, \theta, \varphi) = \frac{GM}{r} \left[1 + \sum_{l, \text{pair}}^{+\infty} \left(\frac{R}{r} \right)^l V_{l,0} \cdot Y_l^0(\theta, \varphi) \right]$$

$$V(r, \theta, \phi) = \sum \kappa_{l,m} \cdot f(r) \cdot P_{l,m}(\cos \theta) \cdot e^{im\phi}$$

Potentiel du champ de gravité -> Géoïde

- Si la Terre était une sphère homogène en rotation, alors sa forme serait celle d'un ellipsoïde, et **de même celle du potentiel de gravité**.
- Les termes de degrés plus grands que 2 sont des **perturbations** qui s'ajoutent à cet ellipsoïde.
- Le potentiel généré par une sphère en rotation (qui devient un ellipsoïde) est appelé **potentiel hydrostatique**. *(Cela parce que si une planète était entièrement liquide, alors la surface de cette planète suivrait exactement cet ellipsoïde.)*
- De fait, la **surface des océans** (liquide) suit exactement le potentiel de gravité. Plus exactement une surface sur laquelle la gravité est constante : une **équipotentielle de gravité**. On nomme cette surface le **Géoïde**.
- Le Géoïde hydrostatique est la surface que les océans suivraient si la Terre n'était qu'une sphère **homogène** en rotation.
- Le **Géoïde non-hydrostatique** est ce qu'il reste une fois que cet effet est pris en compte. *(il serait nul si la Terre était une simple sphère homogène en rotation)*

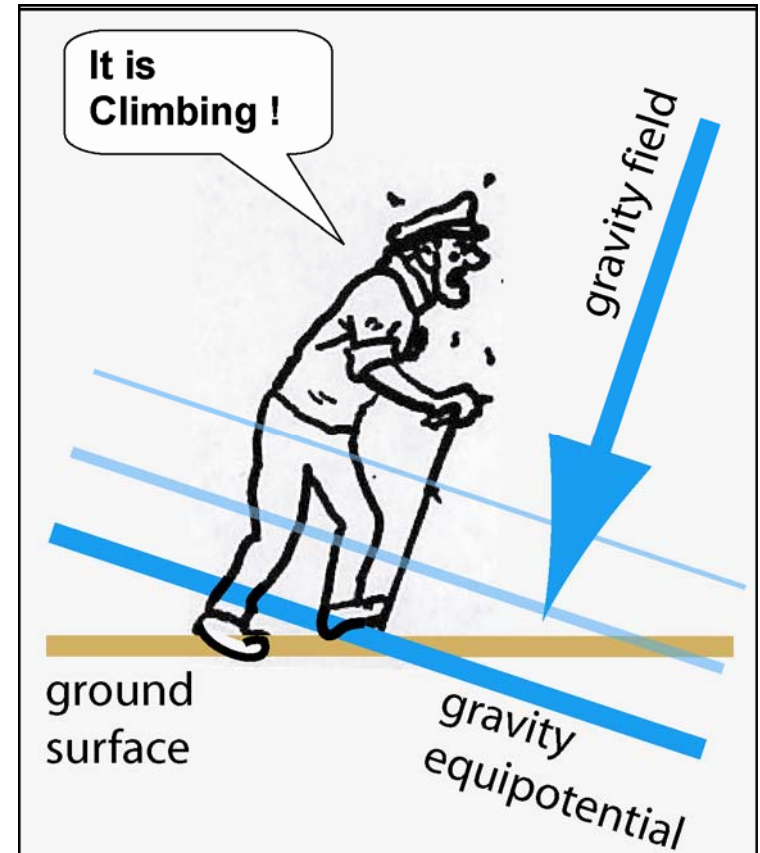
Géoïde : Géoïde Non-hydrostatique, modèle GEM-T1



Les creux sont en bleu, les bosses en rouge. Le Géoïde ondule de +/- ~100 m. Sa forme est celle d'une balle de Tennis.

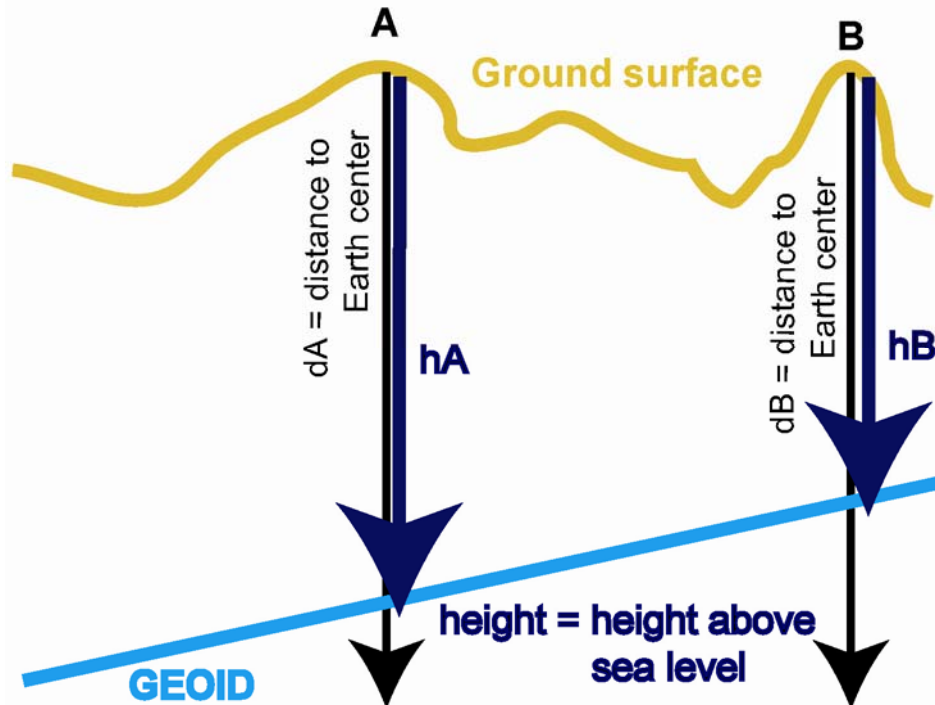
Géοïde : Définition de l'altitude

L'altitude (la hauteur) **n'est pas** un concept purement géométrique (i.e. la distance d'un point à un autre). Elle est définie **par rapport au potentiel de gravité (donc au Géoïde)**



Geoid : Definition of altitude

Altitude (height) **is not** a purely geometrical concept (i.e. distance from one point to the other) it is **defined with respect to the gravity potential**.



2 points **A** and **B** at ground level

One might think their altitude is d_A and d_B

But it is not !!!!

The altitude is the distance to the **geoid** (i.e. the **sea level**) : **h_A and h_B**

If the Geoid is not flat (i.e. at the same distance from the center of the Earth at A and B), **then the altitude changes**

$$G(\theta, \phi) = \sum_{l=0}^{+\infty} \sum_{m=-l}^l \kappa_{l,m} \cdot P_{l,m}(\cos \theta) \cdot e^{im\phi}$$

Geoid : Spectral contains

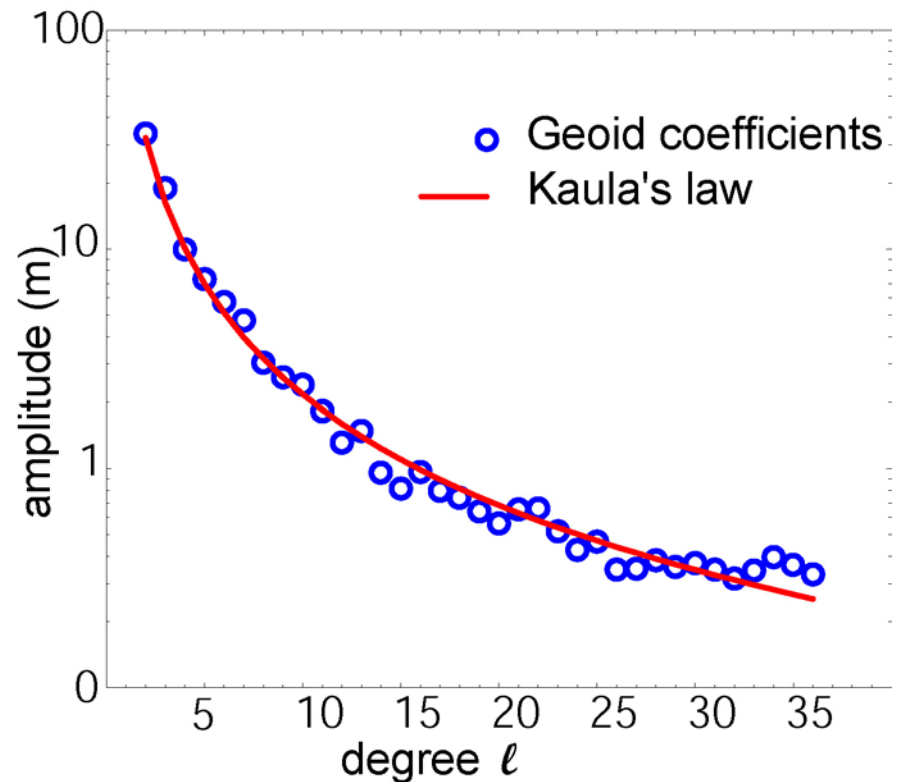
Spectrum of a field = amplitude of coefficients at a given degree l of the spherical harmonic decomposition

$$S_l = \sqrt{\sum_{m=-l}^l (\kappa_{l,m})^2}$$

The spectrum of the Geoid obeys a power law :

$$S_l \sim 1/l^2$$

It is called **Kaula's law**



$$V(\theta, \phi) = \sum_{l=0}^{+\infty} (R/r)^l \sum_{m=-l}^l \kappa_{l,m} \cdot P_{l,m}(\cos \theta) \cdot e^{im\phi}$$

Altitude dependent spectral contains :

The potential V at altitude $r-R$ is attenuated by a coefficient :

$$(R/r)^l$$

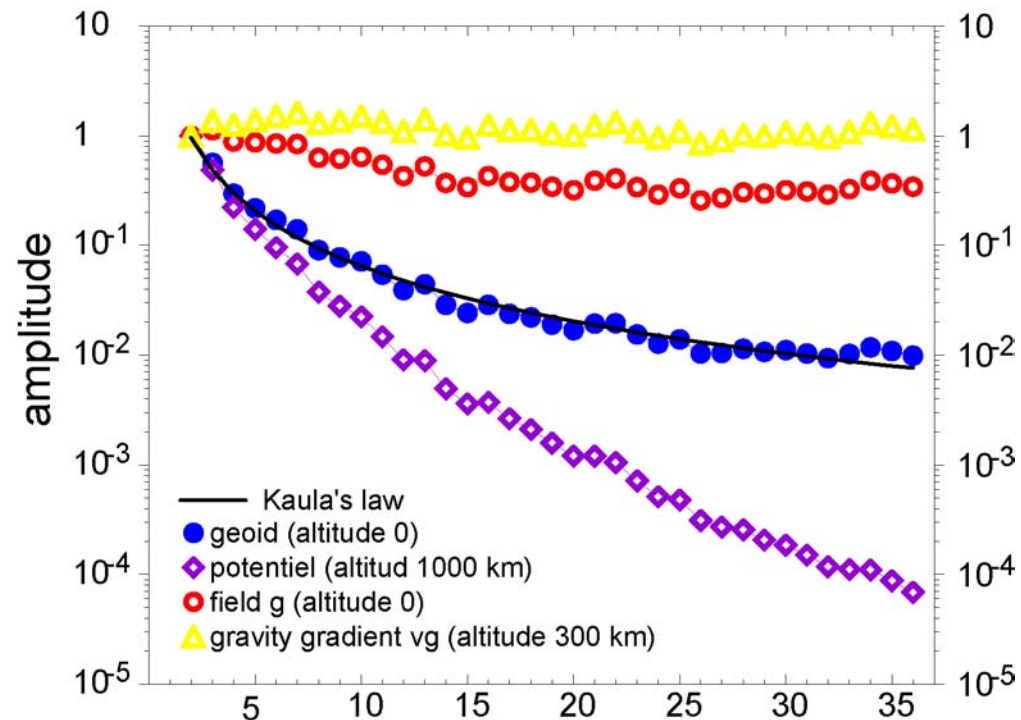
The gravity field g is the derivative of the gravity potential :

$$g = \partial V / \partial r$$

$$\Rightarrow g_l \sim V_l \times l$$

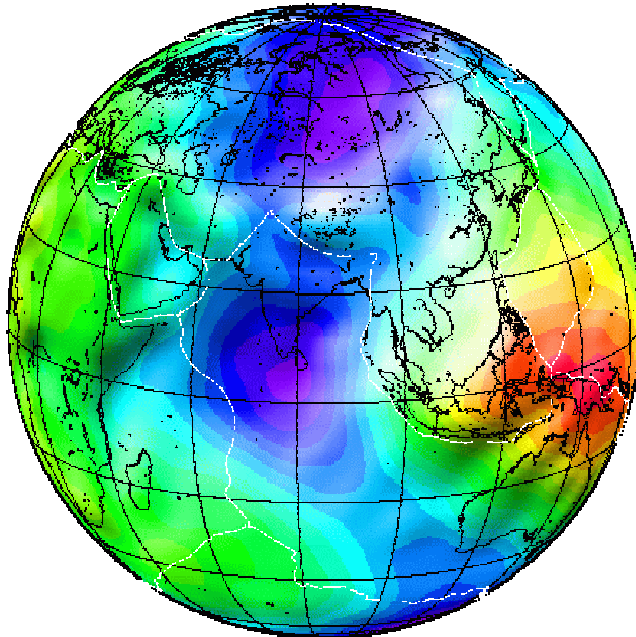
The gradient of the gravity field ∇g is the 2nd derivative of the gravity potential :

$$\Rightarrow \nabla g_l \sim V_l \times l^2$$



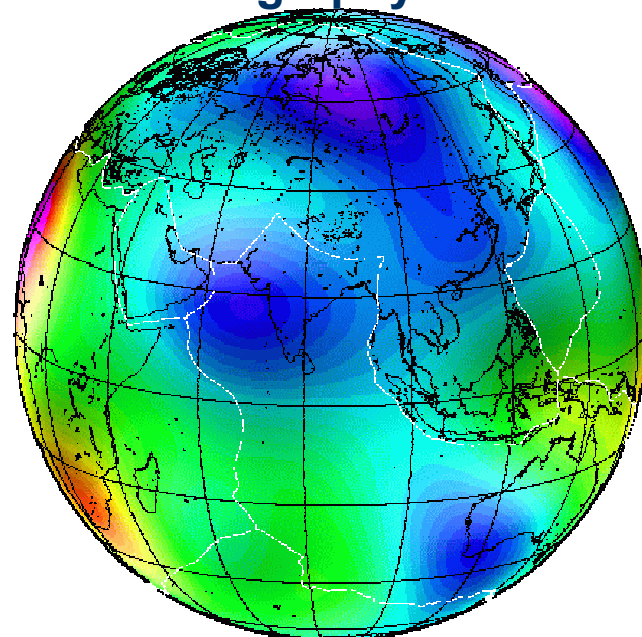
Origin of the Geoid : density anomalies

Geoid over India



Blue=low gravity
Red = high gravity

Seismic tomography in the mantle



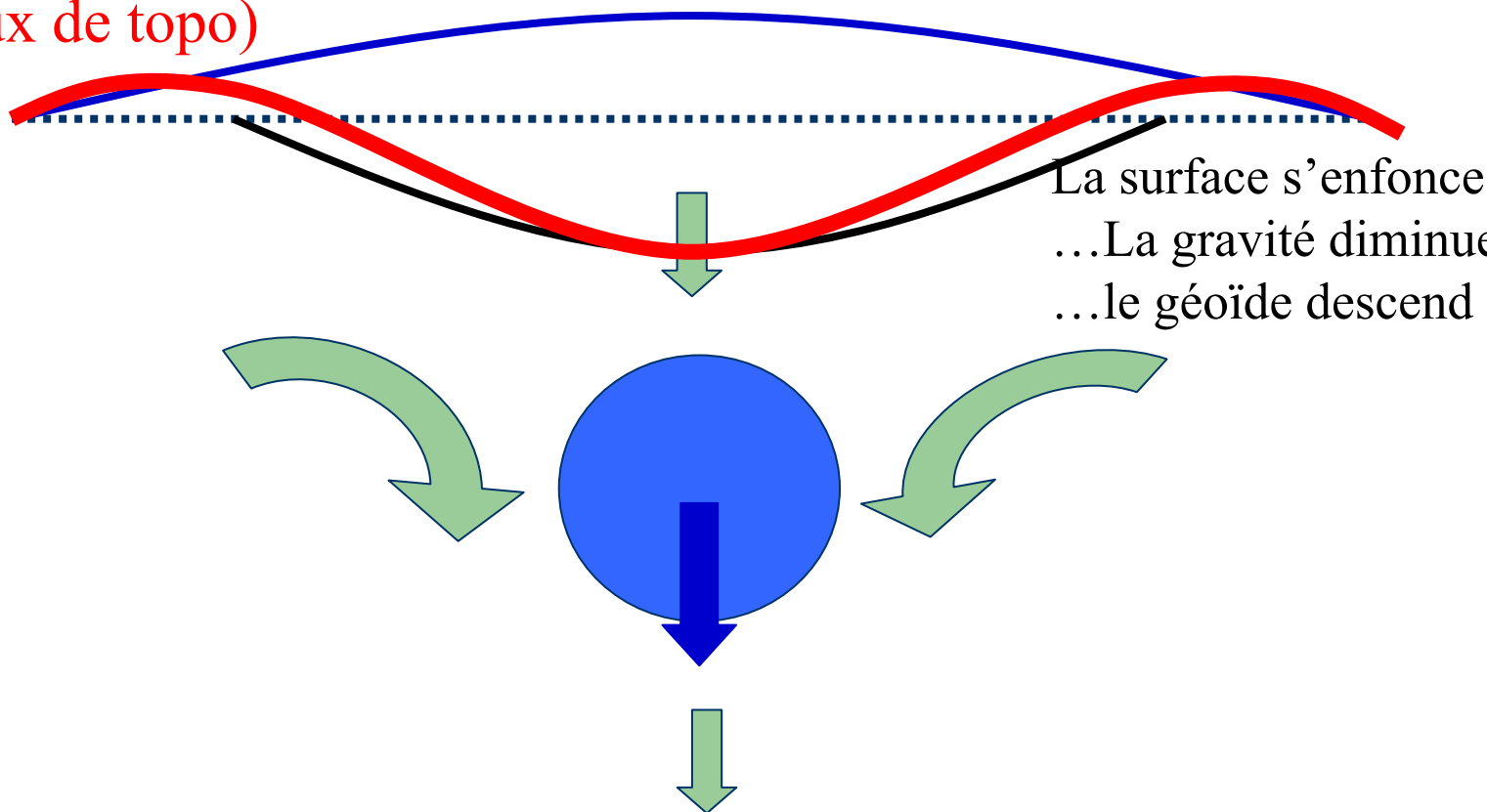
Blue="cold"="more dense"
Red = "hot" = "less dense"

It is very clear that long wavelength Geoid lows are associated to cold and dense material in the mantle ???????

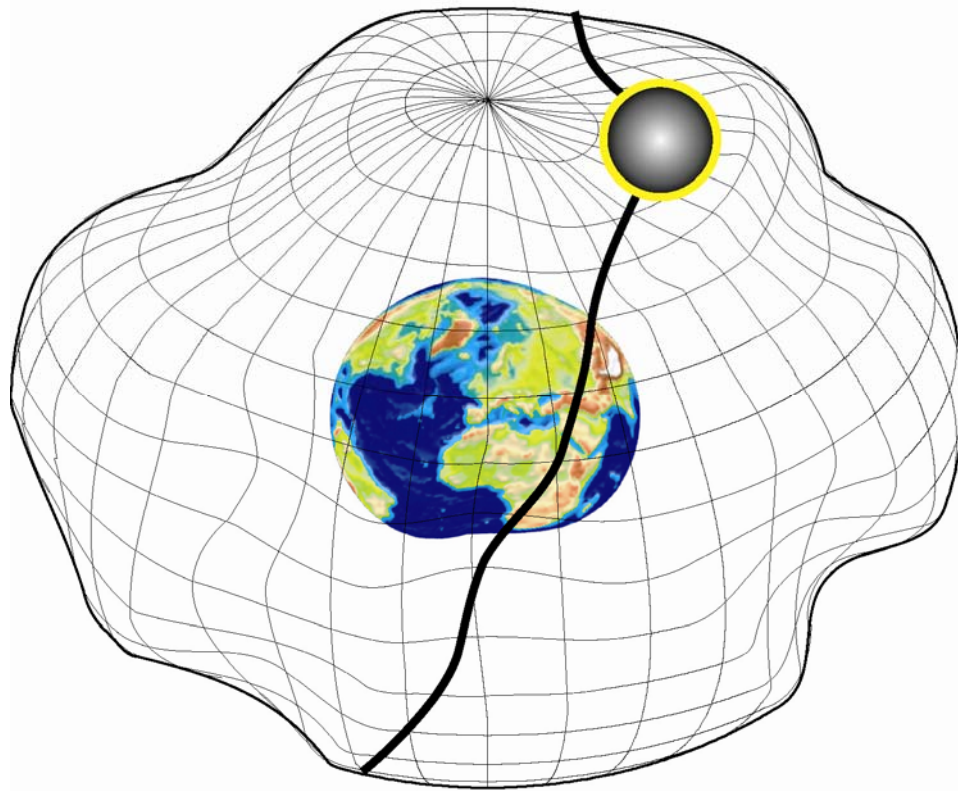
Explication : Géoïde généré par une masse

Géoïde total (masse +
creux de topo)

Géoïde du à la masse



Measurement of the Geoid : spatial geodesy



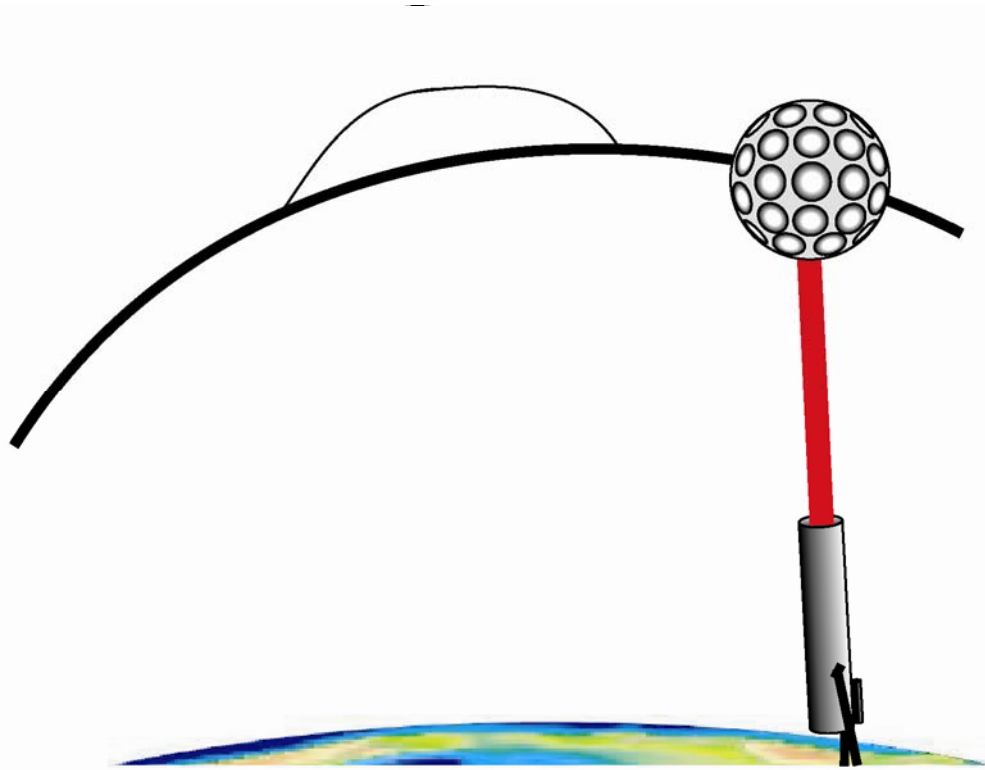
A satellite orbiting around the Earth will be sensitive to gravity: Its motion is such that the rotation force exactly equilibrates the gravity forces.

If the gravity is stronger (i.e. the gravity potential higher), then the satellite will have to orbit a little bit farther away from the earth (to increase the rotation force, and remain in equilibrium)

Conclusion : an orbiting satellite will follow exactly the Gravity potential !

measuring the satellite orbit will give us the gravity potential (i.e. the Geoid)

SLR : Satellite Laser Ranging



A High power laser fires on the satellite

The impulse comes back, so the travel time is measured.

Given the speed of light (**C**), one can compute the distance from laser station to satellite :

$$L = \Delta t \times C$$

Measuring distances along the orbit give the shape of this orbit, I.E., the shape of the gravity potential

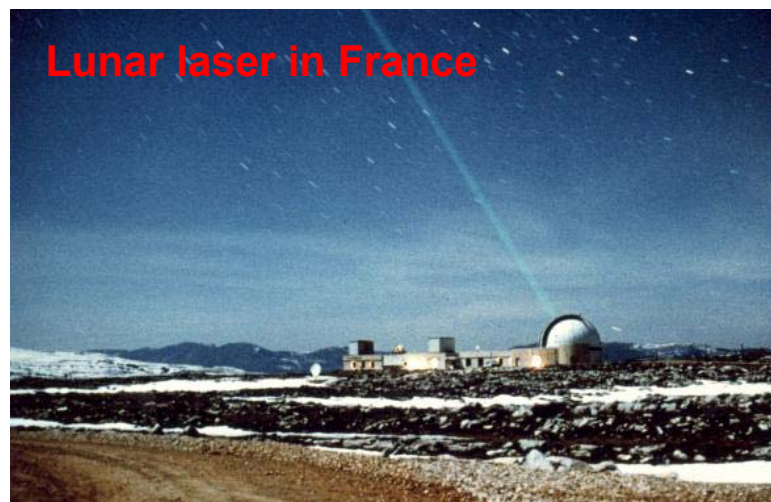
SLR : Satellite Laser Ranging



MOBLAS7 in South Africa



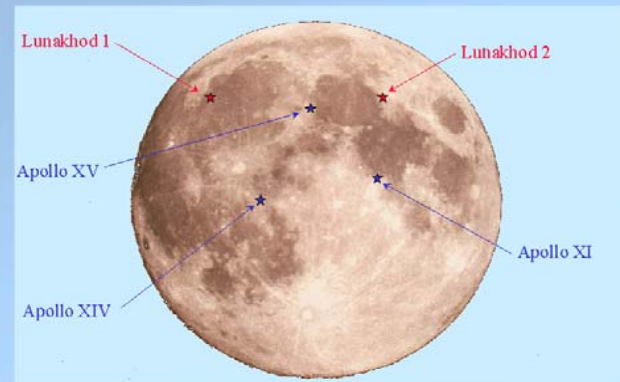
Lunar laser in France



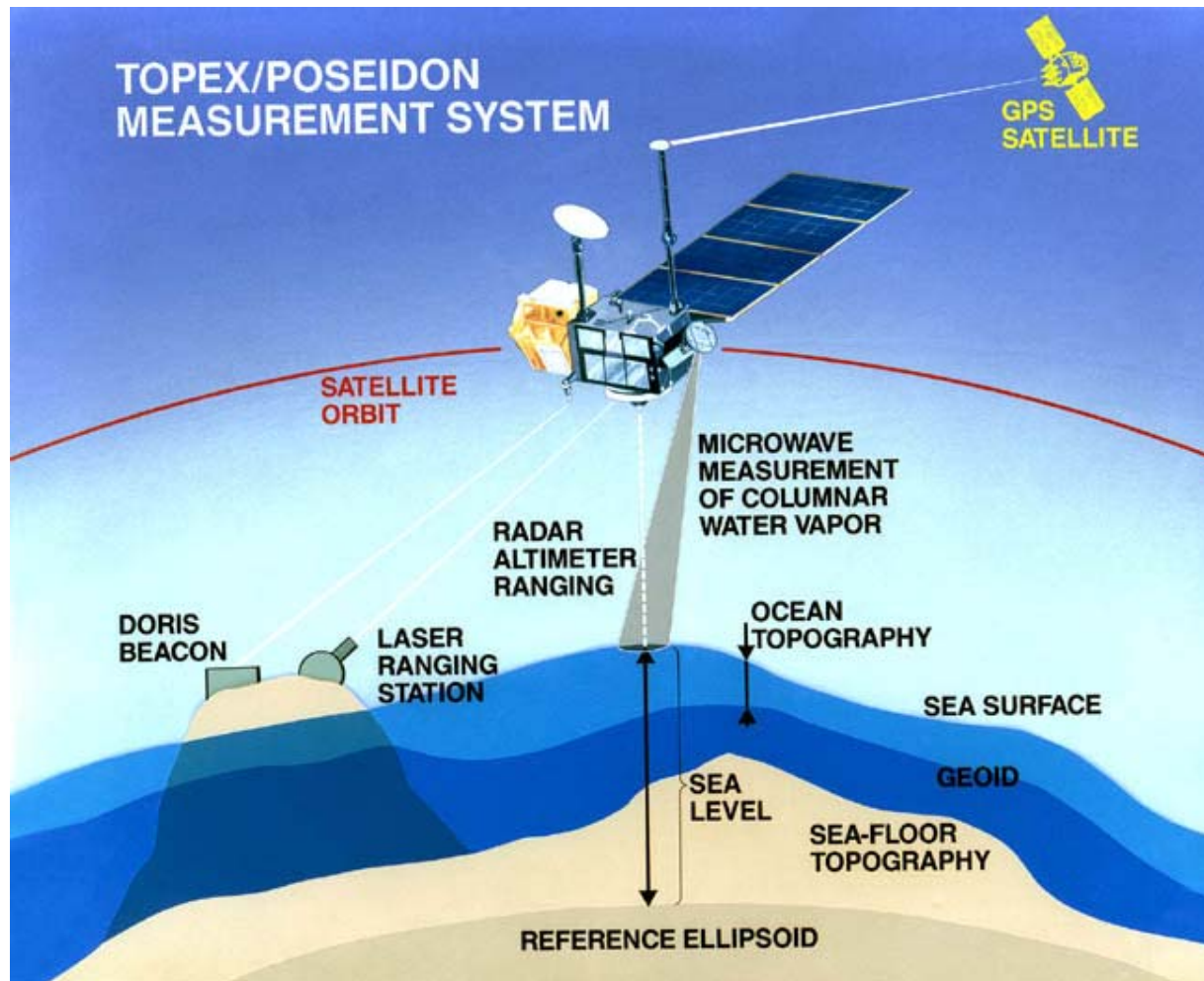
SLUM in France



Laser reflectors on the Moon



Satellite altimetry : principle



A satellite radar measures the distance between the satellite and the surface of the sea

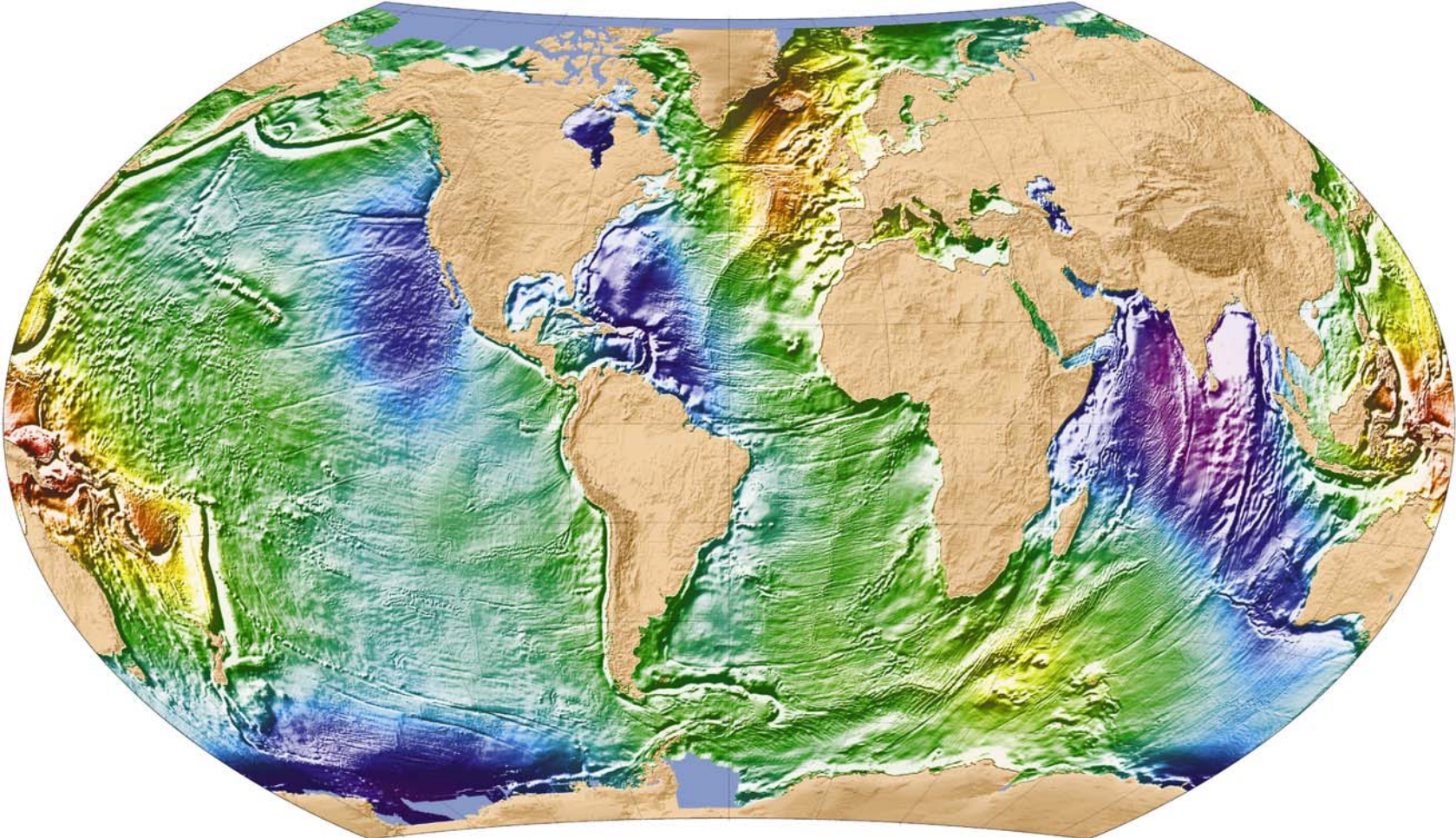
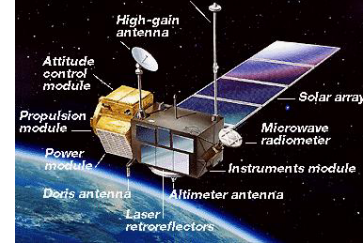
In average (not considering waves, tides and oceanic currents) **the sea surface is the Geoid**

ds = distance satellite to center of Earth

h = distance satellite to sea surface (measured)

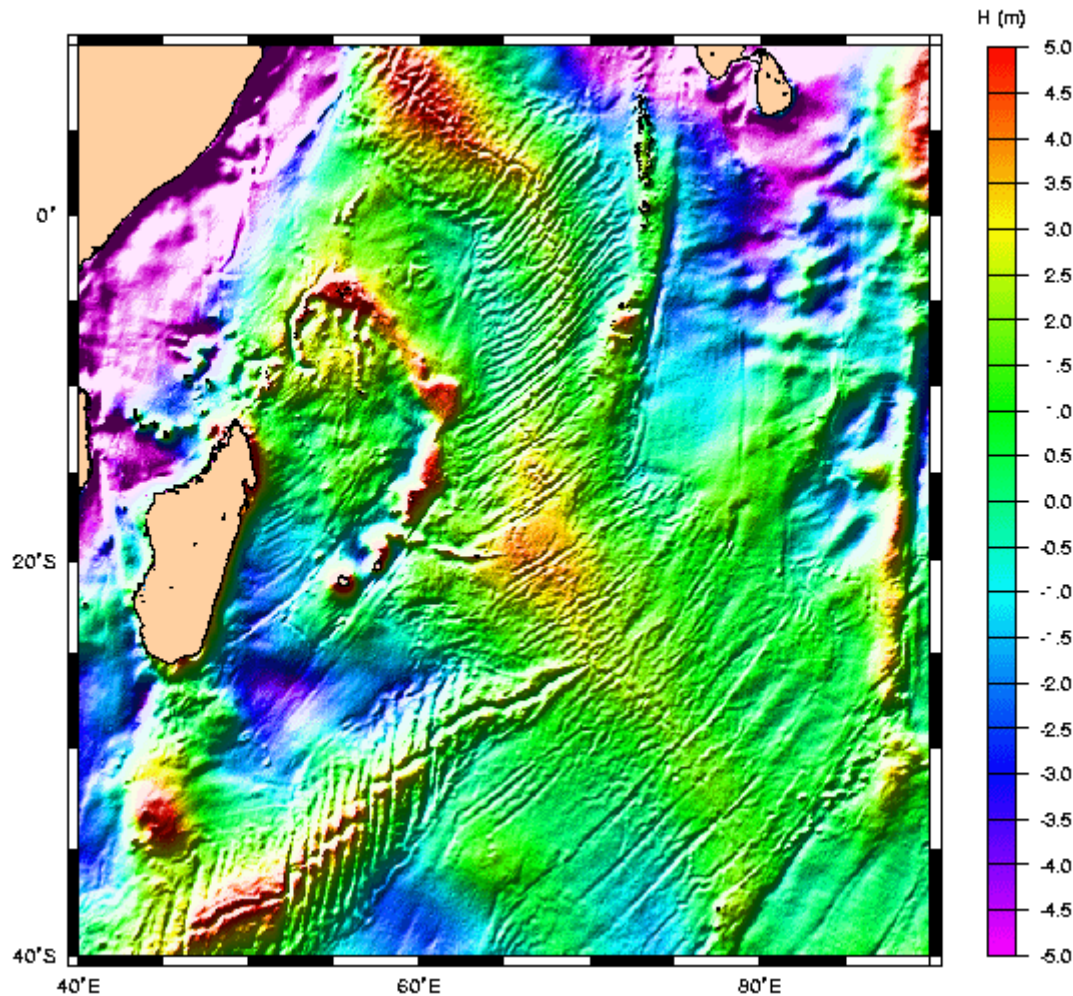
$$\text{Geoid} = ds - h$$

Satellite altimetry



The result is a high resolution map of the Geoid on 70% of the earth surface

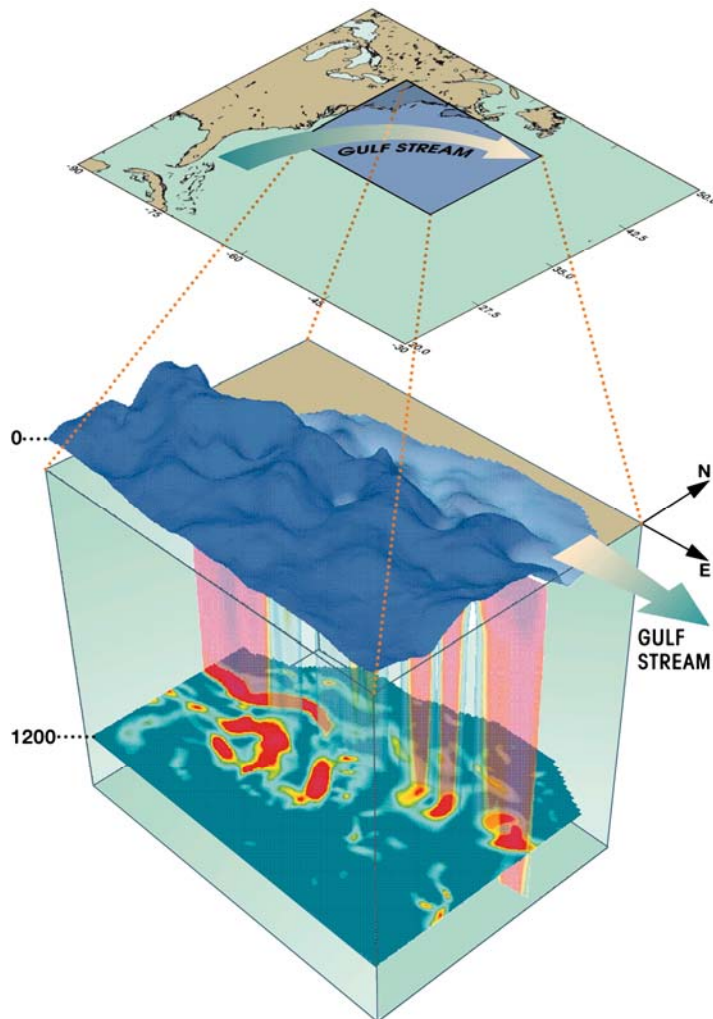
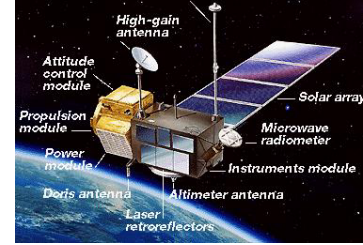
Satellite altimetry



A zoom of the oceanic Geoid shows that we see in detail **short wavelength** gravity anomalies

These come from density anomalies at the surface of the sea bottom. They are **ridges, sea mounts, transform faults, etc...**

Satellite altimetry

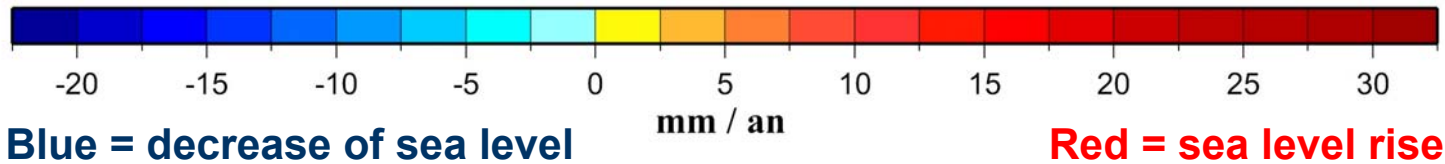
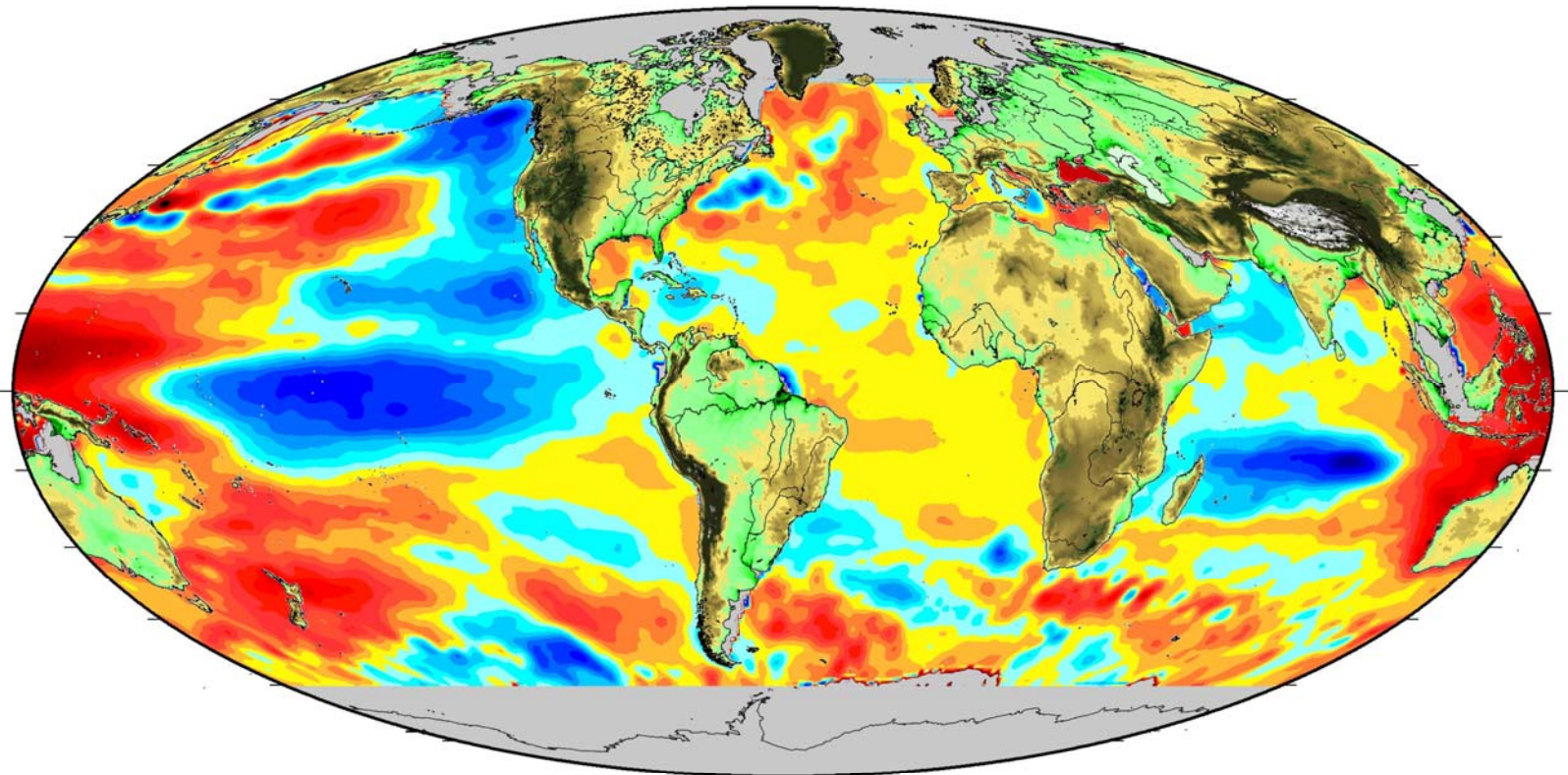
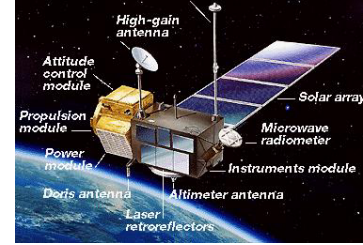


An anomaly of the sea surface can also be related to **water anomaly**

The precision of current altimeter allow to map swells of no more than **10 cm**.

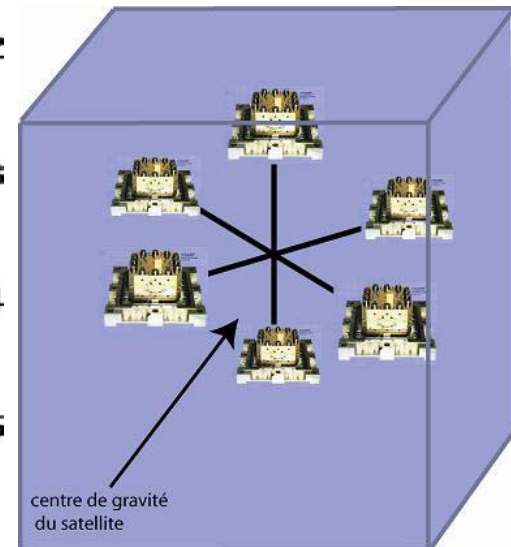
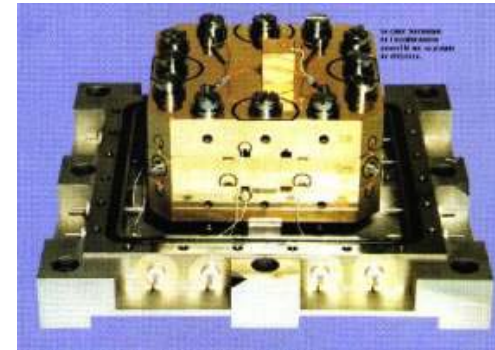
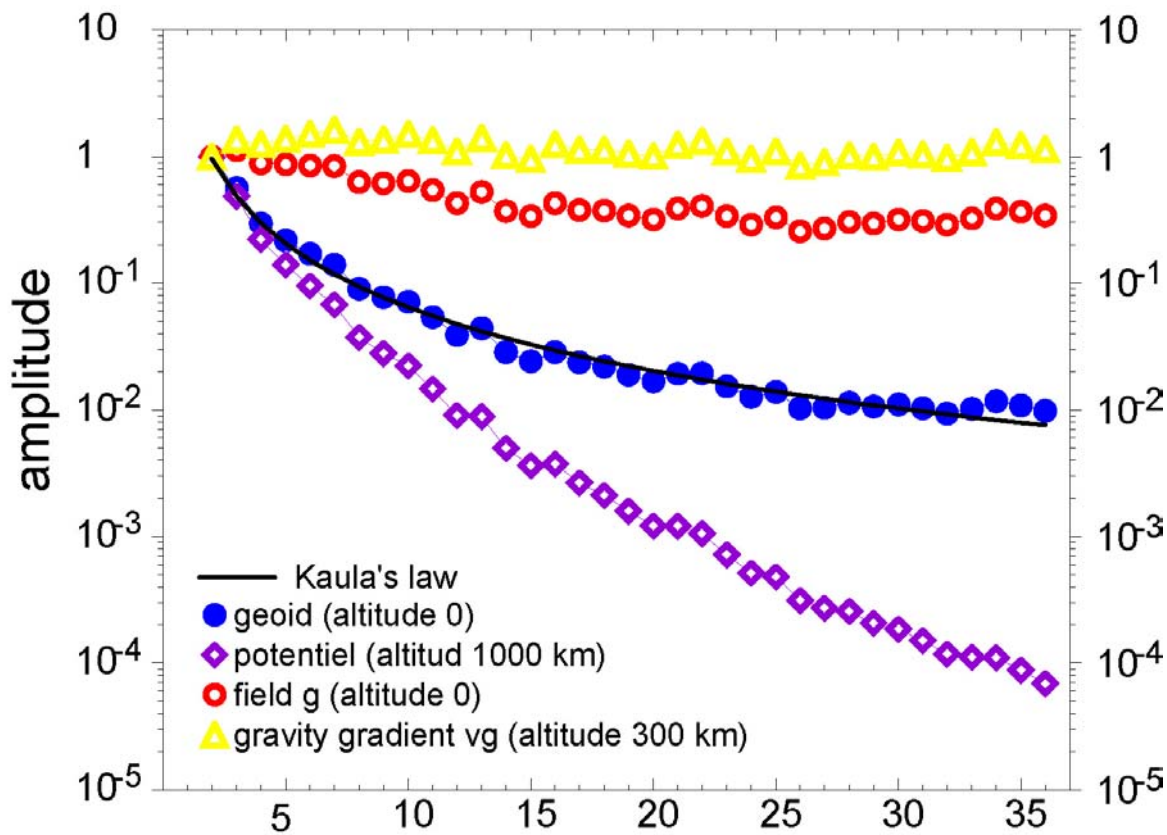
Doing this, we can trace oceanic currents like the **Gulf Stream**

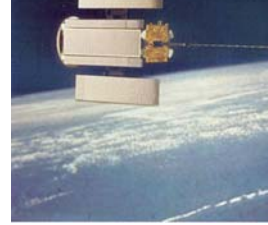
Satellite altimetry: Sea level variation



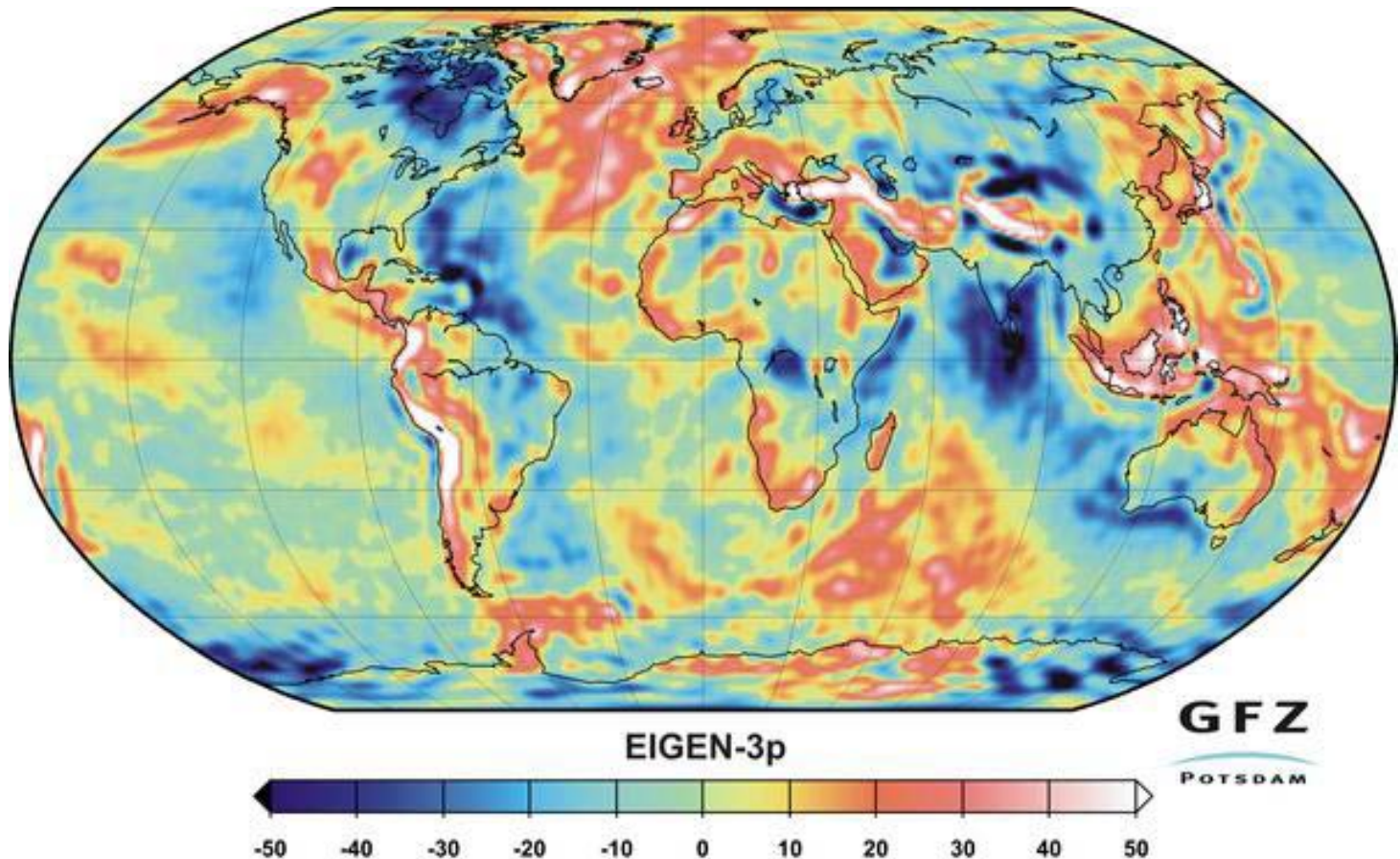


Satellite gradiometry





Satellite gradiometry

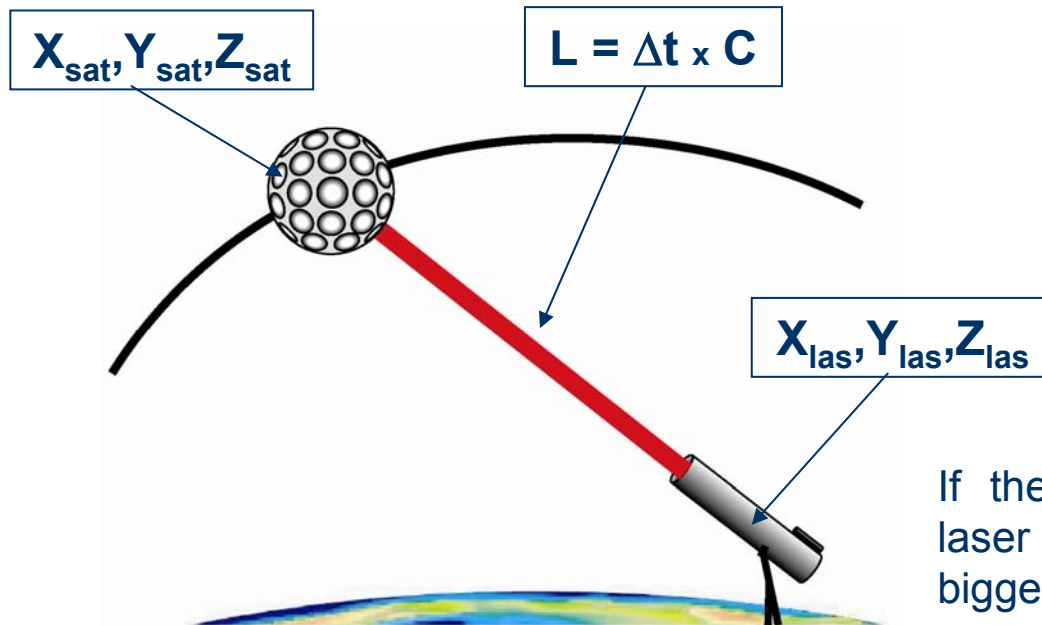




Earth surface deformation

Satellite Laser Ranging

High energy laser firing at satellites enable to determine the position of the satellite and then the Geoid assuming the station position is know. On reverse, assuming one knows the satellite position (i.e. the earth gravity field), then by measuring the satellite-station distance one can determine the station position. The time is measured with a precision of about **0.1ns to 0.3 ns** (3.10^{-10} sec), which give a precision of about **3 to 10 cm** on the measured length, hence on the station position.



$$X_{las} = X_{sat} - L_x$$

$$Y_{las} = Y_{sat} - L_y$$

$$Z_{las} = Z_{sat} - L_z$$

$$pos_{las} = pos_{sat}(t_i) - L(t_i)$$

With : t_i = time of i^{th} measurement along the orbit

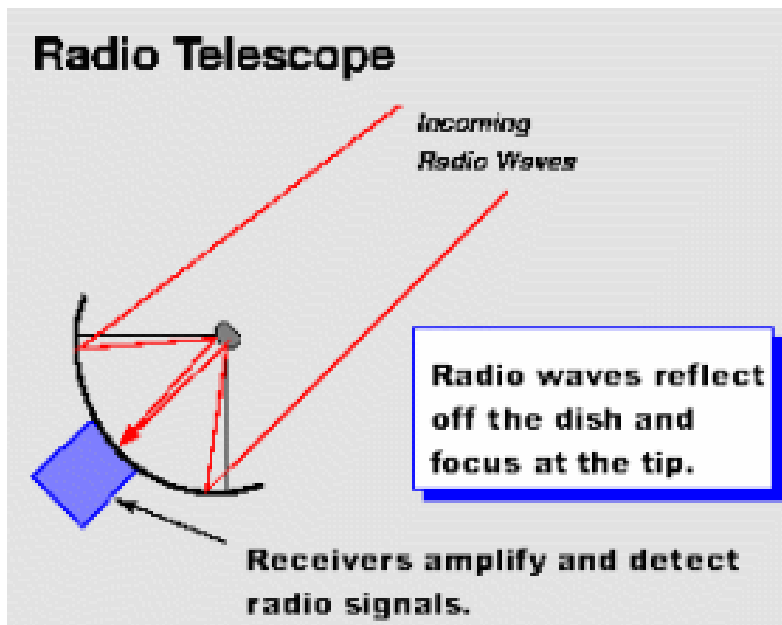
If the earth surface deforms, then the laser station moves. If this motion is bigger than a few cm, then the measurement detects it !

Earth surface deformation

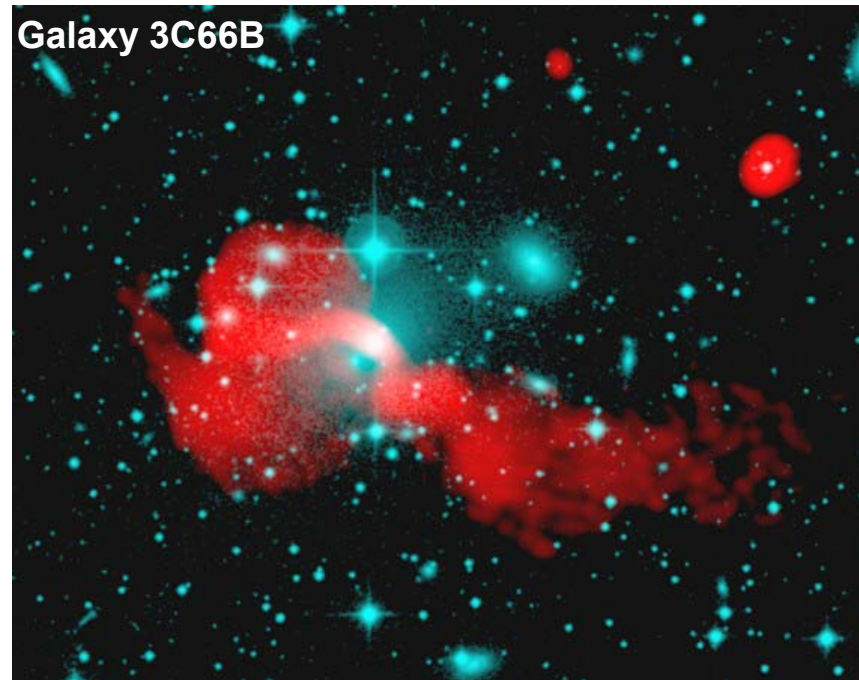


Radio Telescope principle

Radio telescopes are used to study naturally occurring radio emission from stars, galaxies, quasars and other astronomical objects between wavelengths of about 10 meters (30 megahertz [MHz]) and millimeter (300 gigahertz [GHz]). At wavelengths longer than about 20 centimeters (1.5 GHz) irregularities in the ionosphere distort the incoming signals. Below wavelengths of a few centimeters absorption in the atmosphere becomes increasingly critical. the effective angular resolution and image quality is limited only by the size of the instrument.



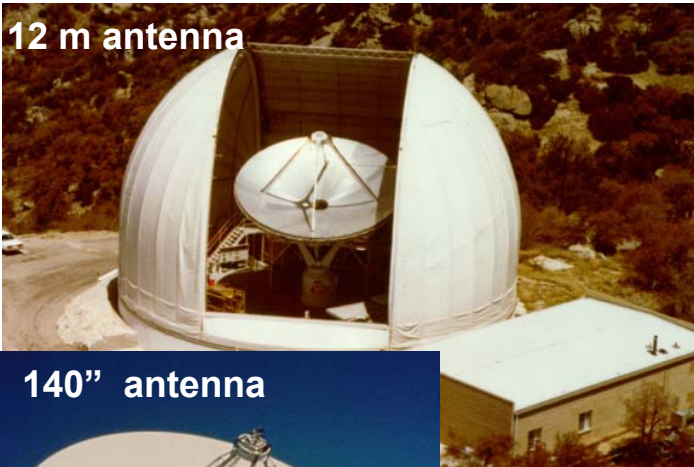
Galaxy 3C66B



Bigger antennas



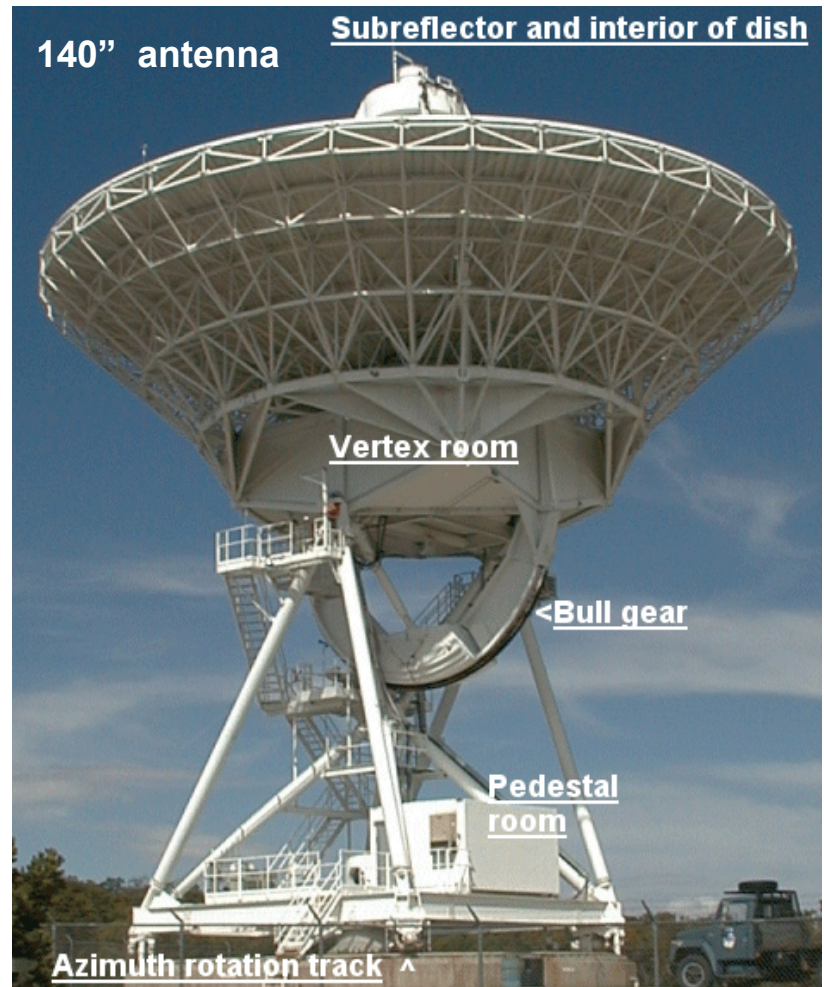
12 m antenna



140'' antenna



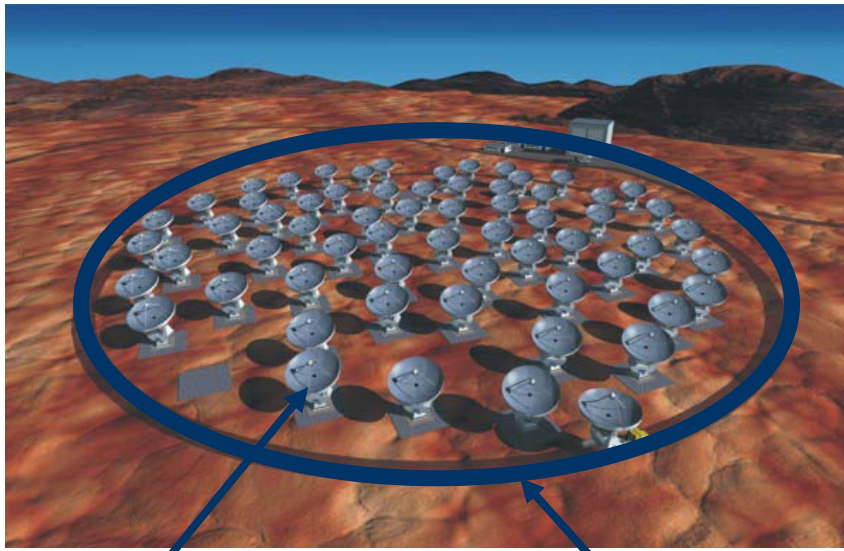
140'' antenna



Very Large Base Interferometry (VLBI)



It is extremely difficult to built antennas bigger than 20-30 meters diameter...
But, one **single large** mirror (or antenna) can be replace by **many sma** mirrors (or antenna). The size of the image will be equivalent. Thus, an array of small antennas make a **virtual** big antenna of equivalent size the size of the array.



Single small antenna

virtual antenna

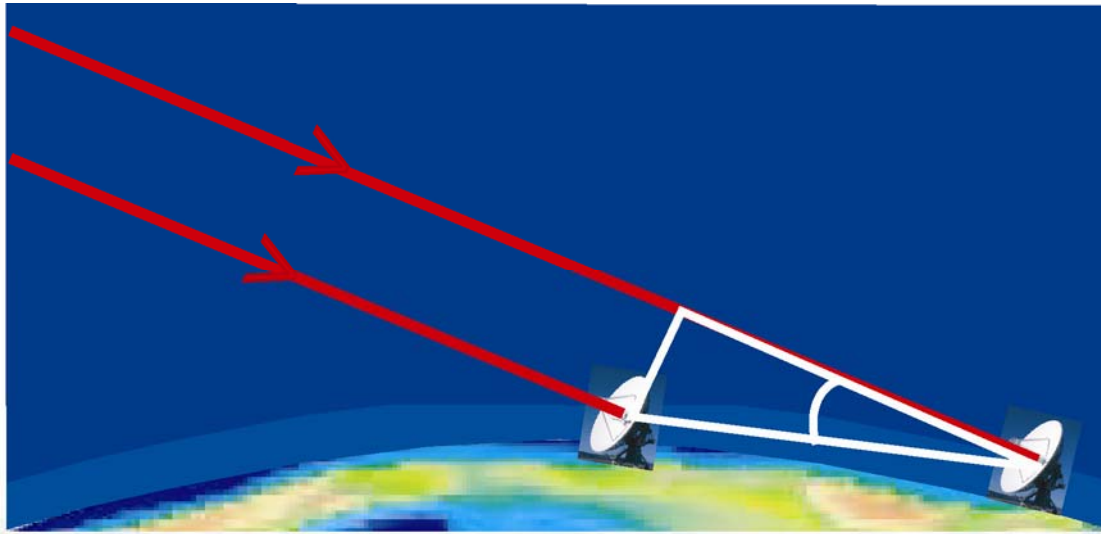


Very Large Base Interferometry (VLBI)



One can reconstruct a precise image of the observed object, knowing precisely the distances between the individual antennas. If these distances are not well known, then the image is fuzzy.

Again, reversing the problem, focusing a known image allow to determine the distances between stations.



The radio wavelength λ arrives at first antenna at time t , and at the second antenna at time $t + \Delta t$.

The additional distance is $\Delta t \cdot c$

Which we can easily convert into distance between stations (knowing the angle = difference in latitude)

The obtained precision is around **1 millimeter** !

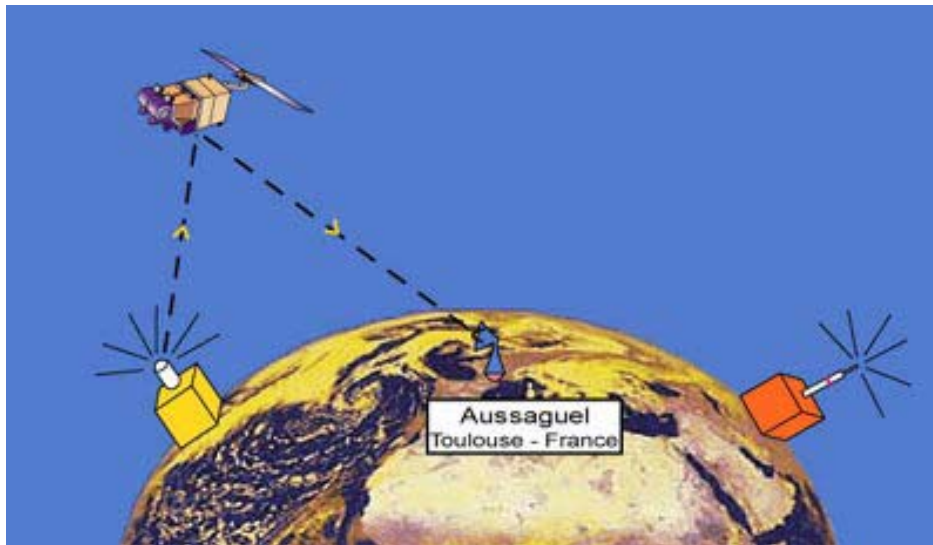
DORIS (Doppler system)



A wavelength is broadcasted by a ground station with a given frequency. A satellite is receiving this signal. Because the satellite is moving, the frequency it receives is shifted. This is the Doppler effect.

For a velocity $\mathbf{v}$, the frequency ν will be shifted by a quantity equal to $\mathbf{v} \cdot \mathbf{v}/c$

The complete formula for $\mathbf{v}$ not // to line of view is :
$$\nu' = \nu \frac{1 - \cos \phi \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$



For a satellite velocity and position are linked by the Keplerian equation of its orbit.

Thus, measuring the Doppler shift allows to determine the Station to Satellite distance

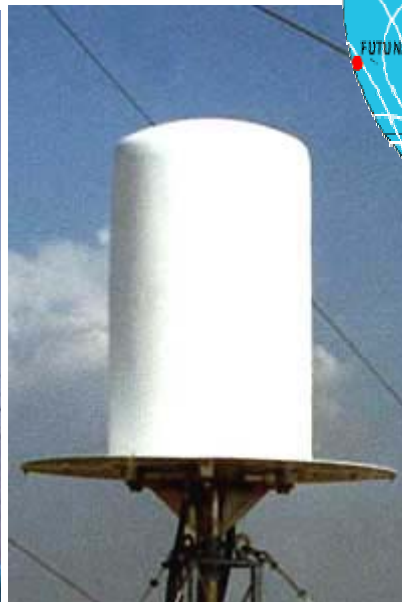
DORIS (Doppler system)



The obtained precision on station position is around 1-3 cm



DORIS GLOBAL network
~60 stations covering the whole Globe



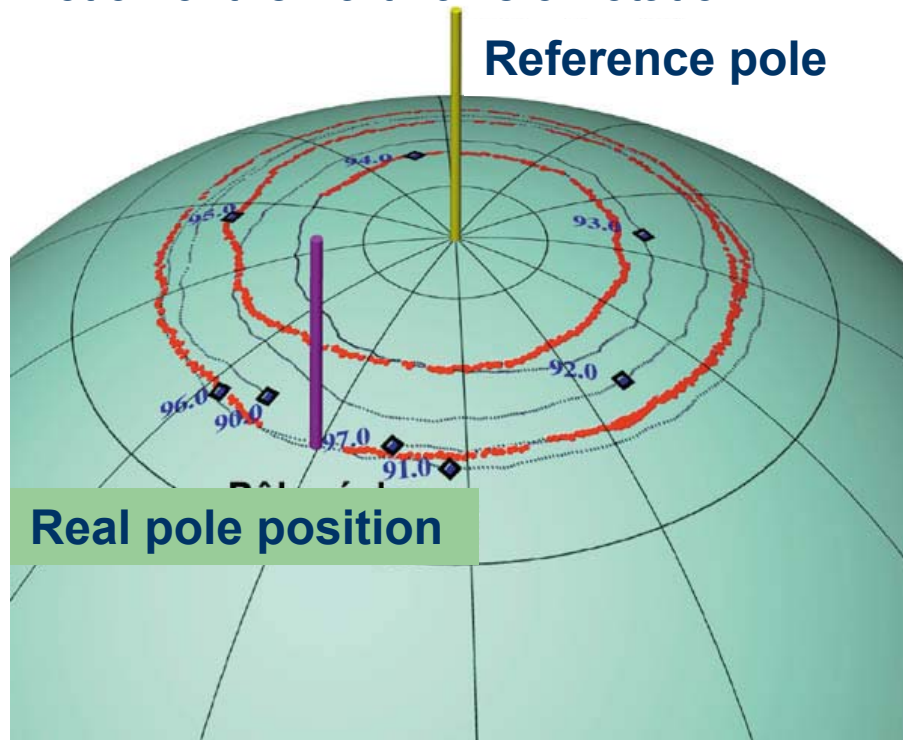
DORIS beacons

DORIS (Doppler system)

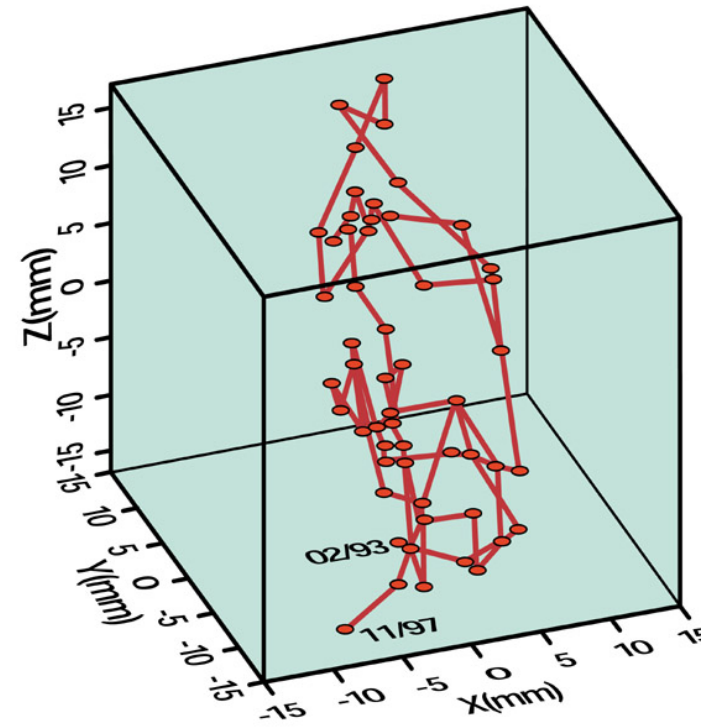


DORIS allow to detect motion of stations but also the motion of the whole network (as polyhedron) in space. Thus we can determine the **oscillations of planet Earth**. These oscillations have a complex frequency contains from Milankovitch period (26 000 years) to Chandler Wobble (400 days) and daily adjustments due to atmospheric loads

Motion of the Earth axis of rotation



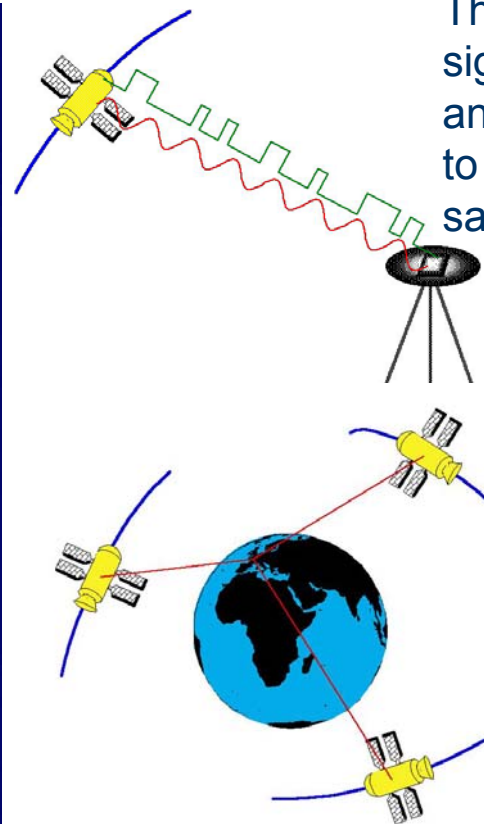
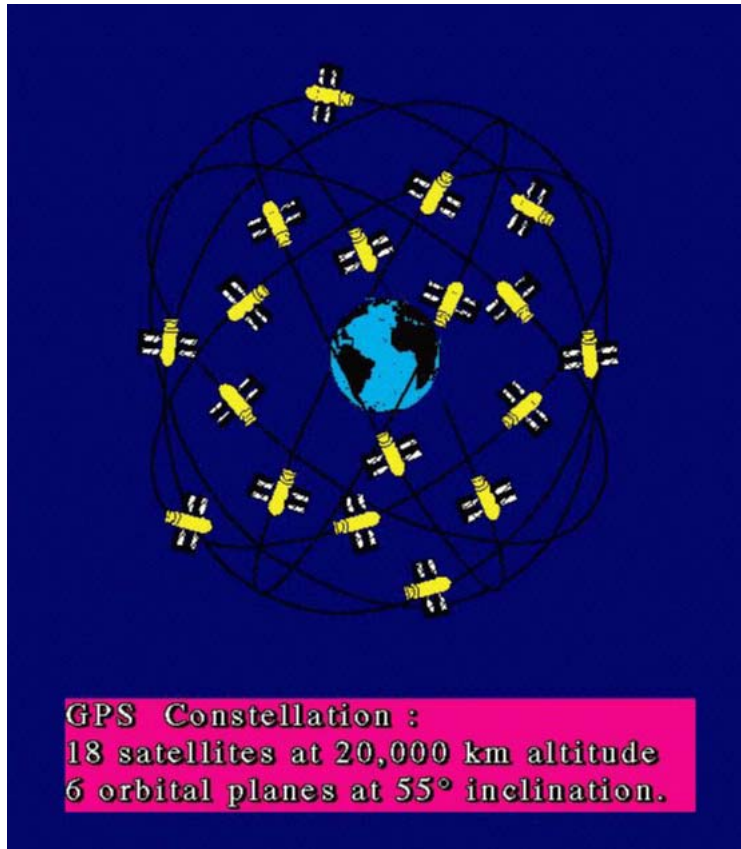
Motion of the Earth gravity center



GPS (Global Positioning System)



GPS was created in the 80s' by the US Department of Defense for military purposes. The objective was to be able to get a precise position anywhere, anytime on Earth.



The satellites send a signal, received by a GPS antenna. Again, this allows to measure the distance satellite to antenna

With at least 3 satellites visible at the same time, we can compute instantaneously the station position. The precision can be as good as 1 millimeter